The Symbol of a Markov Semimartingale

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Introduction

Introduction and Overview

For Lévy processes it is a well known fact that there is a one-to-one correspondence between the elements of this class of processes and the so called continuous negative definite functions in the sense of Schoenberg (cf. [6]) $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$. The connection between these concepts is given by

$$\mathbb{E}^x e^{i(X_t - x)'\xi} = e^{-t\psi(\xi)}.$$

In particular it is known that to every continuous negative definite function ψ there exists a corresponding Lévy process $(X_t)_{t\geq 0}$. Several properties of the process can be expressed in terms of analytic properties of its characteristic exponent ψ (see e.g. [19]). Within the class of (universal) Markov processes, Lévy processes are those which are stochastically continuous and homogeneous in time and space. From the perspective of stochastic modeling the last point is a rather strong restriction since it means that the process 'behaves the same' on every point in space and time. Therefore, it is an interesting question if there exists a function, which is somehow similar to the characteristic exponent of a Lévy process, for a larger class of Markov processes. A class to start with is the one of (nice) Feller processes, i.e. Feller processes with the property that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of their generator. In the investigation of these processes, a family of continuous negative definite functions $\xi \longmapsto p(x,\xi)$, $(x \in \mathbb{R}^d)$ shows up in the Fourier representation of the generator

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} p(x,\xi)\widehat{u}(\xi)\, d\xi \qquad \text{for } u \in C_c^\infty(\mathbb{R}^d). \tag{1}$$

The function $p : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$, which is locally bounded in both variables and continuous negative definite in the second one, is called the symbol of the Feller process. For a Lévy process we obtain $p(x,\xi) = \psi(\xi)$, i.e. the symbol of a Lévy process is its characteristic exponent and it does not depend on x. In any other case the symbol is (like the process) state space dependent.

Unlike in the Lévy case, not every symbol $p(x,\xi)$ corresponds to a (Feller) process. One direction of the research in this area investigates the question if there exists such a process for a given symbol, i.e. if the operator A, defined on a suitable subspace of $C_\infty(\mathbb{R}^d)$ (e.g. the test functions $C_c^\infty(\mathbb{R}^d)$), can be extended to a Feller generator. Possible approaches to this question are via the Hille-Yosida theorem or via the martingale problem. For a survey on the different methods we refer the reader to Section 3 of [32] and the references given there. Another direction of the research deals with the connection between a process and its symbol. R. L. Schilling has shown in a series of papers (see [56], [54] and [55]) that if the growth condition

$$\sup_{x\in\mathbb{R}^d} |p(x,\xi)| \leq c \cdot (1 + \|\xi\|^2) \qquad \text{for } \xi \in \mathbb{R}^d \tag{2}$$

is fulfilled, the symbol can be used to give criteria for the conservativeness and the transience/recurrence of the process. Furthermore, one obtains bounds for the Hausdorff dimension of the sample paths. It is an interesting fact that, although semimartingale characteristics are used more often in the literature, on some occasions the symbol appears to be the more natural object to study, i.e. the criteria can be written in an elegant way by using the symbol. For the interplay between symbol and characteristics see Section 6.1.

The starting point of our considerations is the following probabilistic formula (cf. [54])

$$p(x,\xi) = -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t-x)'\xi}-1}{t} \tag{3}$$

which can be used to calculate the symbol of nice Feller processes, which fulfill the growth condition (2). This formula can be found as well in the survey paper [32], where the authors proposed the following program: *study the process through the symbol.* The present thesis should be seen in the tradition of this program. One of the main questions we are dealing with is, whether the above formula can be used to generalize the notion of the symbol to a larger class of processes and which class this might be. In this context we introduce so called Itô processes (in the sense of [13]). These are (universal) Markov processes which are semimartingales with respect to every $\mathbb{P}^x$ ($x \in \mathbb{R}^d$) having characteristics of the following form:

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\, ds \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega))\, ds \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy)\, ds. \end{aligned} \tag{4}$$

To deal with this class of processes we need concepts and results of different parts of the theory of stochastic processes. The most important facts of the so called 'general theory of processes' are stated without proofs in the last section of this introductory chapter. Universal Markov processes with their connections to semigroup theory and concepts like the infinitesimal generator are treated in Chapter 1. In the literature, there are different notions which generalize the infinitesimal generator. We prove some results on the relationship between the different concepts in Section 1.2. Thereafter Feller processes and their important subclass of Lévy processes are introduced.

In the second chapter we deal with semimartingales, Itô calculus and stochastic differential equations (SDEs). Some important concepts like the characteristic triplet of a semimartingale, which is a generalization of the Lévy triplet, are introduced as well as the notions of the square- and the angle-bracket. Although most of the results in this chapter are well known, we give new proofs for some of them, which are more adapted to our point of view. In particular we show that the solution of a Lévy driven SDE

$$\begin{aligned} dX_t &= \Phi(X_t)\, dZ_t \\ X_0 &= x \end{aligned} \tag{5}$$

is a nice Feller process, if the coefficient Φ is bounded. Furthermore, we prove that this result is in the following sense best possible: if the solution of an SDE of this type is a Markov process, the driving term has to be Lévy. For SDEs driven by Hunt processes we show that the vector consisting of the solution and the driving term is a Markov process and that this result has again a converse.

In Section 3.1 we establish the result that every (nice) Feller process is a semimartingale and even an Itô process. From a practical point of view this means that we cannot only integrate with respect to a Feller process, but we even know how to calculate this integral. Furthermore, this inclusion shows that the class of Itô processes is a candidate for which the generalized symbol could be introduced. Parts of Section 3.2 can be seen as a simplified version of [13]. The results of this interesting but rather technical paper are used to obtain a different perspective on our results in Section 3.1 by using the so called extended generator, which was introduced by E. B. Dynkin (see [15]).

We use the probabilistic formula (3) to define a symbol for an Itô process $X = (X_t)_{t\geq 0}$ in Chapter 4. If the differential characteristics $(\ell, Q, N(\cdot, dy))$ are bounded, this works nicely. If not, we use the following idea: instead of analyzing the original process, we stop X as soon as it leaves an arbitrary (but fixed) compact set K containing a neighborhood of the starting point of X. What we obtain is the symbol

$$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t^T - x)'\xi} - 1}{t} \tag{6}$$

which exists always in the case of (finely) continuous differential characteristics. Furthermore, the limit does not depend on the choice of the compact set K and coincides with the limit (3) in the case of bounded differential characteristics. The limit we obtain is

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y) \right) N(x, dy).$$

For nice Feller processes this symbol coincides with the analytic symbol we encountered in the Fourier representation of their generator (1). We have thus established a generalization of the symbol from nice Feller processes with bounded differential characteristics to Itô processes with unbounded differential characteristics.

In Chapter 5 we return to the SDEs and prove some structural results for the symbols of their solutions. In Section 5.1 the Lévy driven SDE (5) is considered. We obtain that the symbol of the solution is $p(x,\xi) = \psi(\Phi(x)'\xi)$, where ψ is the characteristic exponent of the driving Lévy process and Φ is the coefficient of the SDE. As a by-product we prove that to every symbol of this kind, with bounded Φ, there exists a corresponding Feller process. In Section 5.2 we turn to a more general case. The driving process is now itself an Itô process and the solution is no longer Markovian. However, the bivariate process consisting of the solution and the driving term is an Itô process and admits a symbol, which we calculate (see Theorem 5.4). The most general case is considered in the third section of this chapter. Itô processes can, in principle, be obtained as solutions of SDEs which are of the following kind:

$$\begin{aligned} X_t &= x + \int_0^t \ell(X_s)\, ds + \int_0^t \sigma(X_s)\, d\tilde{W}_s \\ &+ \int_0^t \int_{\mathbb{R}} k(X_{s-}, z) 1_{\{|k(X_{s-},z)|\leq 1\}} \left(\tilde{\mu}(ds, dz) - ds\ N(dz) \right) \\ &+ \int_0^t \int_{\mathbb{R}} k(X_{s-}, z) 1_{\{|k(X_{s-},z)|> 1\}}\ \tilde{\mu}(ds, dz). \end{aligned}$$

We calculate the symbol for solutions of this type of SDE in order to obtain the general structure of the symbol of an Itô process.

In the sixth Chapter we present some applications of the symbol. In the first section we emphasize the close relationship between the following concepts:

$$\text{(extended) generator} \leftrightarrow \text{symbol} \leftrightarrow \text{semimartingale characteristics}$$

Roughly speaking: if we calculate the symbol, we know immediatly the generator on $C_c^2(\mathbb{R}^d)$, the extended generator on $C_b^2(\mathbb{R}^d)$ and the semimartingale characteristics. This allows us to connect the question, if there exists a Feller process (or an Itô process) associated with a given symbol with Jacod's semimartingale problem (see [37] (12.9)) and with solvability problems of certain SDEs. In particular results in one of these directions can be transformed to the other settings.
For the symbols one can introduce so called indices. W. E. Pruitt did this on the level of Lévy processes and R. L. Schilling generalized this definition to nice Feller processes. In Section 6.2 we introduce the indices for Itô processes and give the following nice characterization for the index β_∞^x:

$$\beta_\infty^x = \limsup_{\|\eta\|\to\infty} \sup_{\|y-x\|\le 2/\|\eta\|} \frac{\log|p(y,\eta)|}{\log\|\eta\|}.$$

Furthermore, we prove that in the case of a Lévy driven SDE with bounded Φ (cf. Section 5.1) the index of the solution is equal to the index of the driving term. This allows us some conclusions on the p-variation of the solution and the behavior of the maximum process: let β_∞^ψ denote the index of the driving Lévy process, which does not depend on the starting point x. We obtain for the solution X that the strong p-variation is $\mathbb{P}^x$-a.s. finite (for every $x \in \mathbb{R}^d$) on every compact time-interval $[0,T]$, if $p > \beta_\infty^\psi$. Furthermore, we show that

$$\begin{aligned}
\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* &= 0 \quad \text{for every } \lambda > \beta_\infty^\psi \\
\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* &= \infty \quad \text{for every } \lambda < \beta_\infty^\psi
\end{aligned}$$

holds for the maximum process $(X_\cdot - x)_t^* := \sup_{0\le s\le t}\|X_s - x\|$.

In Chapter 7 we calculate the symbol of the COGARCH process which was introduced in [42] and which is used to model financial data. Having our results of Section 6.1 in mind it is straightforward to write down the generator and the semimartingale characteristics of this process. Section 6.2 is devoted to a classical example of the solution of an SDE, namely the Ornstein-Uhlenbeck process.

Within our investigations we have obtained the following inclusions for different classes of stochastic processes:

$$\text{Lévy} \subset \text{(nice) Feller} \subset \text{Itô} \subset \text{Hunt semimartingale} \subset \text{Hunt}$$

In the appendix a collection of examples and counterexamples which give some insight into the relationship between these classes of stochastic processes is enclosed. In particular we show that every inclusion in the above diagram is strict.

Notation

Most of the notation which is used in this thesis is (more or less) standard. Whenever we introduce a new notation we write ':='.

We write $\mathbb{N} := \{1, 2, ...\}$ for the positive integers starting with 1 and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+ := [0, \infty[$ and $\mathbb{C}$ denote the complex numbers.
The minimum of two real numbers as well as the pointwise minimum of two real valued functions is denoted by $a \wedge b := \min\{a, b\}$, while we write $a \vee b := \max\{a, b\}$ for the maximum.

A vector x in $\mathbb{R}^d$ is a column vector. For a transposed vector or matrix we use $'$. In particular for vectors x, y the scalar product is denoted by $x'y$. The components of the vector are denoted by $x^{(j)}$ $(j = 1, ..., d)$; the components of a $d \times n$ matrix Φ by Φ^{jk} $(j = 1, ..., d,\ k = 1, ..., n)$. Occasionally we will use a capital letter for the matrix and small letters for the components: $Q = (q^{jk})_{1 \le j \le d, 1 \le k \le n}$. In the context of finite dimensional vector spaces we write $\|\cdot\| := \|\cdot\|_{\ell^2}$ for the Euclidean norm. If other norms are used we usually write $\|\cdot\|_{\ell^p}$ for $p \in [1, \infty]$. $|\cdot|$ denotes the ℓ^1-norm or the absolute value in the one-dimensional case. For the ball of radius r we write: $\overline{B_r(x)} = \{y \in \mathbb{R}^d : \|y - x\| \le r\}$.

The indicator function on an arbitrary space is defined as follows:

$$1_A(y) := \begin{cases} 1 & y \in A \\ 0 & y \notin A. \end{cases}$$

For functions 'increasing' always means 'non-decreasing'. The positive part of a real valued function f is denoted by $f^+(x) := f(x) \vee 0$. The sup-norm on a function space is denoted by $\|\cdot\|_\infty$. For a compact set K the sup-norm on K is $\|\cdot\|_{K,\infty}$. To avoid complicated notations we write $\int_{y \neq 0}$ for $\int_{\mathbb{R}^d \setminus \{0\}}$. ∂_j denotes the j-th partial derivative of a differentiable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$. The second derivative is denoted by $\partial_j^2 := \partial_j \partial_j$. In the context of Taylor's theorem we write for a multiindex $\alpha \in \mathbb{N}_0^d$:

$$\partial^\alpha := \partial^{\alpha^{(1)}} ... \partial^{\alpha^{(d)}}.$$

$\mathcal{B}^d$ denotes the Borel σ-field on $\mathbb{R}^d$. In the one-dimensional case we write $\mathcal{B} := \mathcal{B}^1$. More generally for every topological space G, $\mathcal{B}(G)$ denotes the Borel sets on G. For a family of σ-algebras $(\mathcal{F}_i)_{i \in I}$ we denote by $\vee_{i \in I} \mathcal{F}_i := \sigma(\mathcal{F}_i : i \in I)$ the σ-algebra generated by the family $(\mathcal{F}_i)_{i \in I}$.

The space of probability measures on $\mathbb{R}^d$ is denoted by $M^1(\mathbb{R}^d)$. Elements of this space are usually called $\mathbb{P}$ while other measures are often called μ or ν. $\mathbb{E}$, $\mathbb{E}^x$ resp. $\widetilde{\mathbb{E}}$ denotes the (conditional) expectation w.r.t. $\mathbb{P}$, $\mathbb{P}^x$ resp. $\widetilde{\mathbb{P}}$. If two random variables or σ-algebras are independent we use the symbol $\perp\!\!\!\perp$. The convolution of measures is denoted by $\mu * \nu$ and for the Dirac measure in x we write δ_x.

We are dealing with several spaces of functions: $B(\mathbb{R}^d) := B(\mathbb{R}^d, \mathbb{R})$ are the Borel measurable functions on $\mathbb{R}^d$ and $C(\mathbb{R}^d)$ are the continuous functions. Superscripts are used to denote differentiability while subscripts are used to denote other properties: $C^2(\mathbb{R}^d)$ are the functions which are twice continuously differentiable. $C_b(\mathbb{R}^d)$, $C_c(\mathbb{R}^d)$

resp. $C_\infty(\mathbb{R}^d)$ are continuous functions which are bounded, with compact support resp. which vanish at infinity. The combination of upper and lower indices is to be read as follows: $C_b^2(\mathbb{R}^d)$ are the bounded twice continuously differentiable function such that the first and second derivatives are bounded; $C_\infty^2(\mathbb{R}^d)$ is defined alike. A function $f : \mathbb{R} \longrightarrow \mathbb{R}^d$ is called càdlàg if it is right continuous and has finite left limits. It is called càglàd if it is left continuous with finite right limits. We write $D(\mathbb{R}_+, \mathbb{R}^d)$ for the space of all càdlàg functions. We denote by $\mathrm{S}(\mathbb{R}^d)$ the Schwartz space, which is defined as follows (cf. [44] Section III.4): A function $u : \mathbb{R}^d \longrightarrow \mathbb{K}$ is said to be of rapid decrease if, for any $m \in \mathbb{N}$,

$$(1 + \|x\|^2)^m u(x) \to 0 \qquad \text{as} \qquad \|x\| \to \infty.$$

And we set:

$$\mathrm{S}(\mathbb{R}^d) := \{u \in C^\infty(\mathbb{R}^d) : \partial^\alpha u \text{ is of rapid decrease, for every } \alpha \in \mathbb{N}_0^d\}$$

Let $u \in \mathrm{S}(\mathbb{R}^d)$ then

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix'\xi} u(x) \, dx$$

is called the Fourier transform. It is a bijective mapping $\mathrm{S}(\mathbb{R}^d) \longrightarrow \mathrm{S}(\mathbb{R}^d)$ and we have the following inversion formula for $\varphi \in \mathrm{S}(\mathbb{R}^d)$:

$$\check{\varphi}(x) = \int_{\mathbb{R}^d} e^{ix'\xi} \varphi(\xi) \, d\xi.$$

In probability theory $\check{\varphi}$ is often called the characteristic function. In some books on functional analysis a different normalization is used (cf. [33]).

The General Theory of Stochastic Processes

In this section we give a short overview over some definitions and elementary theorems of the so called general theory of processes. The following statements are both, well known and essential for all that follows. The first part follows mainly [36], while the second part - dealing with martingales - is taken from [52].

(0.1) In all that follows let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The 1- respectively d-dimensional Borel-sets are denoted by $\mathcal{B}$ respectively $\mathcal{B}^d$.

(0.2) The **state space** $(E, \mathcal{E})$ is in our investigations always $(\mathbb{R}^d, \mathcal{B}^d)$. We call a mapping $X : \Omega \times \mathbf{I} \longrightarrow E$, where $\mathbf{I}$ is $\mathbb{R}_+$ or an interval in $\mathbb{R}_+$ a **stochastic process** or simply **process**.

(0.3) For every measure μ on $(\mathbb{R}^d, \mathcal{B}^d)$ we write $(\mathcal{B}^d)^\mu$ for the μ-**completion** of $\mathcal{B}^d$ (see e.g. [9] page 2). Now we set $(\mathcal{B}^d)^* := \cap_\mu (\mathcal{B}^d)^\mu$ where the intersection is taken over all finite μ on $(\mathbb{R}^d, \mathcal{B}^d)$. $(\mathcal{B}^d)^*$ is called the σ-algebra of **universally measurable sets** over $(\mathbb{R}^d, \mathcal{B}^d)$.

(0.4) For fixed $\omega \in \Omega$ the function $t \longmapsto X(\omega, t) = X_t(\omega)$ is called a **path** of the process. If $\mathbb{P}$-almost all paths of a process are continuous, we call the process continuous. We use càdlàg, càglàd... in the same way. Later we will deal with a family of measures $(\mathbb{P}^x)_{x \in \mathbb{R}^d}$. In this setting the property (e.g. continuous paths) has to hold a.s. with respect to every $\mathbb{P}^x$ $(x \in \mathbb{R}^d)$.

(0.5) We write $X_{t-}(\omega) := \lim_{s \uparrow t} X_s(\omega)$ for the left-hand side limit of the path $X_\cdot(\omega)$ if it exists.

(0.6) If X is a càdlàg process, we write

$$\Delta X_t(\omega) := X_t(\omega) - X_{t-}(\omega)$$

for the **jump process** associated to X, if X_{t-} exists. In addition we set $X_{0-}(\omega) := 0$ and obtain $\Delta X_0 = X_0$.

(0.7) We write

$$X_t^* := \sup_{s \le t} |X_s|$$

and $X^* := \sup_{t \ge 0} |X_t|$.

(0.8) A subset of $\Omega \times \mathbb{R}_+$ is called a **random set**. Notice that 1_A is a process. For the projection of A on Ω we write $\pi(A) = \{\omega : \exists t \ge 0 \text{ such that } (\omega, t) \in A\}$. The ω-**slice** of A is denoted by: $A_\omega = \{t \ge 0 : (\omega, t) \in A\}$. A is called **thin** if A_ω is countable for every $\omega \in \Omega$.

(0.9) Let S and T be two mappings: $\Omega \longrightarrow \overline{\mathbb{R}}_+$. The **stochastic interval** $]]S, T]]$ is the random set $\{(\omega, t) : S(\omega) < t \le T(\omega), t \ge 0\}$. We define analogously the stochastic intervals: $[[S, T]]$, $[[S, T[[$ and $]]S, T[[$. Finally we write $[[T]]$ instead of $[[T, T]]$.

(0.10) A **filtration** of $(\Omega, \mathcal{F})$ is a family $\mathbb{F} := (\mathcal{F}_t)_{t \ge 0}$ of sub-σ-fields of $\mathcal{F}$ which is increasing (i.e. $s \le t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$). It is called **right-continuous**, if $\mathcal{F}_s = \cap_{t > s} \mathcal{F}_t$.

(0.11) We write $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t \ge 0}$ for the filtration generated by a process: $\mathcal{F}_t^0 := \sigma(X_s : s \le t)$. This is called the **natural filtration**. And we set: $\mathcal{F}_\infty^0 = \vee_{t \ge 0} \mathcal{F}_t^0$.

(0.12) A filtration is called **complete**, if $\mathcal{F}_0$ contains all $\mathbb{P}$-nullsets. We write $\mathcal{F}^{\mathbb{P}}$ for the completion of $\mathcal{F}$ with respect to the probability-measure $\mathbb{P}$. And we use $\mathcal{F}_t^{\mathbb{P}}$ for the σ-field generated by $\mathcal{F}_t$ and the sets of $\mathbb{P}$-measure zero in $\mathcal{F}^{\mathbb{P}}$. By abuse of notation we call the family $\mathbb{F}^{\mathbb{P}} := (\mathcal{F}_t^{\mathbb{P}})_{t\geq 0}$, which is a filtration on $(\Omega, \mathcal{F}^{\mathbb{P}})$, the **completion** of the filtration $\mathbb{F}$.

(0.13) If a filtration is both: complete and right-continuous, we say that it fulfills the **usual hypothesis**.

(0.14) A random set A is called $\mathbb{P}$**-evanescent** if $\pi(A)$ is $\mathbb{P}$-negligible, i.e. $\pi(A) \in \mathcal{F}^{\mathbb{P}}$ and $\mathbb{P}(\pi(A)) = 0$. Two processes X and Y are called **indistinguishable** if the set $\{X \neq Y\}$ is $\mathbb{P}$-evanescent.

(0.15) We say that the process Y is a **modification** of X if

$$\mathbb{P}(X_t = Y_t) = 1 \text{ for every } t \geq 0$$

holds. In general being indistinguishable is the stronger property, but in the case of càdlàg or càglàd processes they are equivalent.

(0.16) The process Y is called a **version** of X if both have the same finite dimensional distributions, i.e. for every finite sequence $t_1 < t_2 < .. < t_k$ in $\mathbb{R}_+$ we obtain

$$\mathbb{P}_{X_{t_1},..,X_{t_k}}(B) = \mathbb{P}_{Y_{t_1},..,Y_{t_k}}(B) \text{ for every } B \in \mathcal{B}^k.$$

(0.17) A process is called **measurable** or $\mathcal{F}$-measurable if it is a measurable function on the space $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$.

(0.18) A process X is called $\mathbb{F}$**-adapted** or simply adapted, if for every $t \geq 0$ the random variable X_t is $\mathcal{F}_t$-measurable.

(0.19) We call a process X $\mathbb{F}$**-progressive** if for every $t \geq 0$ the mapping $X(\omega, s)|_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t])$-measurable. If a process is adapted and càdlàg it is already progressive.

(0.20) We introduce two σ-fields on $\Omega \times \mathbb{R}_+$ which are of fundamental importance: the **optional** σ-field $\mathcal{O}$ is generated by the set of all adapted càdlàg processes; the **predictable** σ-field $\mathcal{P}$ is generated by all càg processes. As $\mathcal{P}$ is generated by the continuous processes as well, we obtain: $\mathcal{P} \subset \mathcal{O}$. A process which is measurable with respect to $\mathcal{O}$ resp. $\mathcal{P}$ is called optional resp. predictable.
On the space $\widetilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ we define $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}^d$ and $\widetilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{B}^d$.

(0.21) A mapping $T : \Omega \longrightarrow \overline{\mathbb{R}}_+$ is called a **stopping time** if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. It is immediate to see that this is equivalent to the fact that the stochastic interval $[[0, T[[$ is optional. If $[[0, T[[\in \mathcal{P}$ we call T a **predictable time**. If the filtration is right continuous the stopping time property is equivalent to the fact that $\{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

(0.22) We denote by X_T the random variable for which $X_T(\omega) = X_{T(\omega)}(\omega)$. It is important to distinguish this from the **process stopped at time T**

$$X_t^T(\omega) := X_{T(\omega)\wedge t}(\omega) = X_t(\omega) \cdot 1_{[[0,T[[}(\omega, t) + X_T(\omega) \cdot 1_{[[T,\infty[[}(\omega, t).$$

(0.23) If T is a stopping time we define

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall t \geq 0\}.$$

This family of sets is a σ-algebra. Let us remark that for $t \geq 0$ the function $T := t$ is a (deterministic) stopping time. In this case $\mathcal{F}_t$ and $\mathcal{F}_T$ coincide and therefore, there is no ambiguity.

(0.24) If T is a stopping time and $A \subset \Omega$ we write T_A for the mapping defined as follows:

$$T_A(\omega) = \begin{cases} T(\omega) & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A. \end{cases}$$

If $A \in \mathcal{F}_T$ this is again a stopping time.

(0.25) An adapted càdlàg process X is called **quasi left continuous** if for any increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ with limit T we have

$$\lim_{n\to\infty} X_{T_n} = X_T \text{ a.s. on } \{T < \infty\}.$$

Note that the exceptional nullset may depend on the sequence $(T_n)_{n\in\mathbb{N}}$.

(0.26) An $\mathbb{R}-$valued stochastic process $X = (X_t)_{t\geq 0}$ with the properties
  (a) X is adapted, i.e. X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.
  (b) Every X_t $(t \geq 0)$ is integrable, i.e. for all $t \geq 0 : \mathbb{E}\,|X_t| < \infty$.
  (c) For $s < t$: $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$
is called a **martingale**. With $\geq$ resp. $\leq$ in (c) we obtain sub- resp. supermartingales.

(0.27) If X is a martingale and the family $\{X_t : t \geq 0\}$ is uniformly integrable we call the martingale **uniformly integrable**. For the class of all of these processes we write $\mathcal{M}$ or $\mathcal{M}(\mathbb{F}, \mathbb{P})$ to emphasize the dependence on the underlying structure.

(0.28) We denote by $\mathcal{H}^p$ $(p > 0)$ the class of all **p-integrable martingales** that is all martingales such that $\sup_{t\geq 0} \mathbb{E}X_t^p < \infty$. Note that $\mathcal{H}^p \subset \mathcal{M}$. Of special interest are the square integrable martingales $\mathcal{H}^2$.

(0.29) A martingale X is **closable**, if there exists a random variable $X_\infty \in L^1$ such that

$$X_t = \mathbb{E}(X_\infty|\mathcal{F}_t) \text{ for every } \ t \geq 0.$$

The process $(X_t)_{t\in\mathbb{R}\cup\infty}$ is then again a martingale which is called **closed**.

(0.30) For a martingale X the following three conditions are equivalent:
  (a) X is closable;
  (b) $X \in \mathcal{M}$;
  (c) $\lim_{t\to\infty} X_t$ exists in L^1-sense.
If these conditions hold, then $X_\infty = \lim_{t\to\infty} X_t$ almost surely. Moreover, if for some $p > 1$, the martingale is bounded in L^p, i.e. $\sup_{t\geq 0} \mathbb{E}\,|X_t|^p < \infty$, then the conditions (a)-(c) are satisfied and the convergence holds in L^p-sense.

(0.31) [Doob inequality] If $X \in \mathcal{H}^2$:

$$\mathbb{E}\left(\sup_{t\geq 0} X_t^2\right) \leq 4 \sup_{t\geq 0} \mathbb{E}(X_t^2) = 4\mathbb{E}X_\infty^2.$$

More general we have for a positive submartingale X, $p > 1$ and q conjugate to p, i.e. $(1/p + 1/q = 1)$:

$$\|X^*\|_{L^p} \leq q \cdot \sup_{t\geq 0} \|X_t\|_{L^p}.$$

(0.32) Let X be a martingale with respect to the filtration $\mathbb{F}$ which fulfills the usual hypothesis. Then there exists a modification of X which is càdlàg.

(0.33) [optional sampling] If X is a martingale and S, T are two bounded stopping times with $S \leq T$, then

$$X_S = \mathbb{E}(X_T | X_S) \text{ a.s.}$$

(0.34) If X is uniformly integrable, the family $\{X_S : S \text{ stopping time}\}$ is uniformly integrable and if $S \leq T$

$$X_S = \mathbb{E}(X_T | \mathcal{F}_S) = \mathbb{E}(X_\infty | \mathcal{F}_S) \text{ a.s.}$$

(0.35) A càdlàg adapted process X is a martingale if and only if for every bounded stopping time T, X_T is in L^1 and

$$\mathbb{E}(X_T) = \mathbb{E}(X_0).$$

(0.36) If X is a martingale and T a stopping time, the stopped process X^T is again a martingale with respect to $\mathbb{F}$.

(0.37) A stochastic process $(\Omega, \mathcal{F}, \mathbb{F}, (X_t)_{t \geq 0}, \mathbb{P})$ is called **simple Markov process** if it is adapted to $\mathbb{F}$ and

$$\mathbb{P}(X_t \in B | \mathcal{F}_s) = \mathbb{P}(X_t \in B | X_s)$$

holds $\mathbb{P}$-a.s. for $s \leq t$ and $B \in \mathbb{R}^d$.

1 Markov, Feller, Lévy

In this chapter we introduce Markov processes and some of their important sub-classes. Our standard references on this topic are [5], [9] and [17]. Within the investigation of Markov, Feller and Lévy processes on several occasions continuous negative definite functions show up. Therefore, we devote the first section to this class of functions.

1.1 Continuous Negative Definite Functions

Before dealing with the different classes of processes we collect a few facts on positive definite functions and continuous negative definite functions. We will encounter this kind of functions several times in the following. The material presented here is taken from the monograph [33] by N. Jacob.

A function $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called **positive definite** if for any choice of $k \in \mathbb{N}$ and vectors $\xi_1, ..., \xi_k$ the matrix $(u(\xi_j - \xi_l))_{1\leq j,l\leq k}$ is positive Hermitian, i.e. for all $\lambda_1, ..., \lambda_k \in \mathbb{C}$ we have

$$\sum_{j,l=1}^{k} u(\xi_j - \xi_l)\lambda_j\overline{\lambda_l} \geq 0.$$

Bochner's theorem says that a function $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ is the Fourier transform of a (positive) Borel measure with finite total mass $\|\mu\|$ if and only if u is continuous, positive definite and such that $u(0) = \|\mu\|$.

Definition 1.1 A function $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called **negative definite** if $\psi(0) \geq 0$ and $\xi \longmapsto e^{-t\psi(\xi)}$ is positive definite for $t \geq 0$.
We write $N(\mathbb{R}^d)$ for the set of negative definite functions on $\mathbb{R}^d$ and $CN(\mathbb{R}^d)$ for the functions which are in addition continuous.

Next we collect some properties of $N(\mathbb{R}^d)$ and $CN(\mathbb{R}^d)$.

Proposition 1.2 ***(properties of (continuous) negative definite functions)***
a) The set $N(\mathbb{R}^d)$ is a convex cone which is closed under pointwise convergence.

b) For $\psi \in N(\mathbb{R}^d)$ it follows that $\overline{\psi}$ and Re ψ belong to $N(\mathbb{R}^d)$, too. Furthermore, we have for every $\xi \in \mathbb{R}^d$

$$\mathit{Re}\ \psi(\xi) \geq \psi(0) \geq 0.$$

c) Any non-negative constant is an element of $CN(\mathbb{R}^d)$.

d) For $\psi \in N(\mathbb{R}^d)$ and $\lambda > 0$ the function $\xi \longmapsto \psi(\lambda\xi)$ belongs to $N(\mathbb{R}^d)$.

e) The set $CN(\mathbb{R}^d)$ is a convex cone which is closed with respect to uniform convergence on compacts.

f) For $\psi_j \in N(\mathbb{R}^{d_j}), j = 1, 2,$ it follows that $\psi\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) := \psi_1(\xi) + \psi_2(\eta)$ defines an element in $N(\mathbb{R}^{d_1+d_2})$.

g) If $\psi \in N(\mathbb{R}^d)$, then $\psi(\cdot) - \psi(0) \in N(\mathbb{R}^d)$.

h) If u is positive definite, then $u(0) - u(\cdot) \in N(\mathbb{R}^d)$.

i) For every negative definite function ψ the function $\sqrt{|\psi|}$ is sub-additive, i.e.

$$\sqrt{|\psi(\xi_1 + \xi_2)|} \leq \sqrt{|\psi(\xi_1)|} + \sqrt{|\psi(\xi_2)|} \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^d.$$

j) For any locally bounded negative definite function $\psi \in N(\mathbb{R}^d)$ there exists a constant c_ψ such that for all $\xi \in \mathbb{R}^d$

$$|\psi(\xi)| \leq c_\psi (1 + \|\xi\|^2),$$

where $c_\psi = 2 \cdot \sup_{\|\eta\| \leq 1} |\psi(\eta)|$.

Remark: A version of Property j), for x-dependent continuous negative definite functions, is shown in Section 3.1.

For our purposes the following Lévy-Khinchine representation of the functions in $CN(\mathbb{R}^d)$ plays a fundamental role.

Theorem 1.3 *The function $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is continuous negative definite if and only if it can be written in the following form:*

$$\psi(\xi) = a - i\ell'\xi + \frac{1}{2}\xi' Q \xi - \int_{y \neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot 1_{\{\|y\|<1\}}(y) \right) N(dy) \qquad (1.1)$$

with $a \geq 0$, $\ell \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ is a positive semidefinite matrix and N is a Lévy measure, i.e. a Borel measure on $\mathbb{R}^d \backslash \{0\}$ such that

$$\int_{y \neq 0} \left(1 \wedge \|y\|^2 \right) N(dy) < \infty.$$

In the literature the Lévy-Khinchine formula (LKF) is often given with different (looking) integral terms. The difference relies mainly on the choice of the so called **truncation function**. In the above theorem we used: $1_{\{\|y\|<1\}}$ which is sort of canonical, but has the disadvantage of not being continuous. A truncation function has the following properties:

(i) $c \in B_b(\mathbb{R}^d)$.

(ii) $c(y) = 1 + o(\|y\|)$ for $\|y\| \longrightarrow 0$.

(iii) $c(y) = O(1/\|y\|)$ for $\|y\| \longrightarrow \infty$.

Switching to a different truncation function c changes the drift coefficient ℓ in the Lévy-Khinchine formula in the following way:

$$\ell_c = \ell + \int_{y \neq 0} y \left(c(y) - 1_{\{\|y\|<1\}}(y) \right) N(dy).$$

We will see that it is advantageous to use the cut-off function of the semimartingale setting (see Section 2.2) in the Lévy-Khinchine formula:

Definition 1.4 Let $R > 0$. We call $\chi = \chi_R : \mathbb{R}^d \longrightarrow \mathbb{R}$ a **cut-off function** if it is measurable and has the following property:

$$1_{\overline{B_R(0)}} \leq \chi_R \leq 1_{\overline{B_{2R}(0)}}. \tag{1.2}$$

With the truncation function $c(y) = \chi(y)$ we obtain

$$\psi(\xi) = a - i\ell'\xi + \frac{1}{2}\xi'Q\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y)\right) N(dy). \tag{1.3}$$

1.2 Markov Processes

There are different concepts of Markov processes in the literature. In the following we will need the notion of a universal Markov process. From the point of view of the introductory section on the general theory we are dealing with a family of processes

$$(\Omega, \mathcal{G}, \mathbb{G}, (X_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}.$$

We call such a family a **(universal) Markov process** if X is adapted to $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ and

$(MP1)$ $\mathbb{P}^x(X_{s+t} \in B \mid \mathcal{G}_s) = \mathbb{P}^{X_s}(X_t \in B)$ $\mathbb{P}^x$-a.s. for every $x \in \mathbb{R}^d$, $B \in \mathcal{B}^d$.

$(MP2)$ The mapping $x \longmapsto \mathbb{P}^x(G)$ is measurable for every $G \in \mathcal{G}$.

(MP1) is called the Markov property. Let us remark that for every fixed $\mathbb{P}^x$ $(x \in \mathbb{R}^d)$ the process $(\Omega, \mathcal{G}, \mathbb{G}, (X_t)_{t\geq 0}, \mathbb{P}^x)$ is a simple Markov process in the sense of (0.37). To avoid technical difficulties we always assume that the Markov process is normal, i.e. $\mathbb{P}^x(X_0 = x) = 1$ for every $x \in \mathbb{R}^d$.

The most common way to construct Markov processes is to start from a **transition function**. It is a family of functions $(P_{s,t})_{s,t\geq 0}$ such that (for every $r \leq s \leq t$)

$(TF1)$ $P_{s,t} : \mathbb{R}^d \times \mathcal{B}^d \longrightarrow [0,1]$.

$(TF2)$ $x \longmapsto P_{s,t}(x, B)$ is $\mathcal{B}^d$-measurable for every $B \in \mathcal{B}^d$.

$(TF3)$ $B \longmapsto P_{s,t}(x, B)$ is a probability measure for every $x \in \mathbb{R}^d$.

$(TF4)$ $P_{r,t}(x, B) = \int_{\mathbb{R}^d} P_{r,s}(y, B) P_{s,t}(x, dy)$ for every $B \in \mathcal{B}^d$.

The last identity is called the Chapman-Kolmogorov equation. We are mainly interested in transition functions which are homogeneous in time, i.e. $P_{s,t}(x, B) = P_{s+h,t+h}(x, B)$ for every $s \leq t$, $h \geq 0$, $x \in \mathbb{R}^d$ and $B \in \mathcal{B}^d$. In this case we write $P_{t-s} := P_{s,t}$ and call the one-parameter family $(P_t)_{t\geq 0}$ a **transition semigroup**. The transition function (resp. semigroup) describes the evolution of a random phenomenon in time. Combining it with a starting measure μ, which is a probability on $(\mathbb{R}^d, \mathcal{B}^d)$ one obtains in the time-homogeneous case:

$$\mathbb{P}_{t_1,\dots,t_n}(B_0, B_1, \dots, B_n) := \int_{B_0} \int_{B_1} \dots \int_{B_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \dots P_{t_1}(x, dx_1)\mu(dx)$$

with $B_j \in \mathcal{B}^d$ for $j = 0, 1, \dots, n$ and $0 \leq t_1 \leq \dots \leq t_n$. This is a projective family of measures. Therefore, by Kolmogorov's extension theorem (see e.g. [40] Theorem 2.2.2) there exists a canonical process

$$(\Omega, \mathcal{G}, \mathbb{G}, (X_t)_{t\geq 0}, \mathbb{P}) = ((\mathbb{R}^d)^{\mathbb{R}_+}, \mathcal{B}((\mathbb{R}^d)^{\mathbb{R}_+}), \mathbb{F}^0, (X_t)_{t\geq 0}, \mathbb{P}^\mu)$$

with these finite dimensional distributions, where $\mathbb{F}^0$ denotes the natural filtration (see 0.10). Setting $\mu := \delta_x$ and $\mathbb{P}^x := \mathbb{P}^{\delta_x}$ for every $x \in \mathbb{R}^d$ one obtains a (universal) Markov process. Let us remark that the converse is also true: starting from a (universal) Markov process, it is possible to derive a transition semigroup by setting: $P_t(x, B) := \mathbb{P}^x(X_t \in B)$ (see [5] Theorem 42.7). Let us emphasize that universal Markov processes are homogeneous in time: to this end we denote $P^y_{s,t}(x, B) := \mathbb{P}^y(X_{s+t} \in B | X_s = x)$. We have to show, that for x in the range of X_s the equation $P^y_{s,t}(x, B) = P_{0,t}(x, B)$ holds for $s, t \geq 0$, $y \in \mathbb{R}^d$ and $B \in \mathcal{B}^d$. We have[1]

$$\begin{aligned}
P^y_{s,t}(X_s, B) = \mathbb{P}^y(X_{s+t} \in B | X_s = x)|_{x=X_s} &= \mathbb{P}^y(X_{t+s} \in B | X_s) \\
&= \mathbb{P}^y(\mathbb{P}^y(X_{t+s} \in B | \mathcal{G}_s) | X_s) \\
&= \mathbb{P}^y(\mathbb{P}^{X_s}(X_t \in B) | X_s) \\
&= \mathbb{P}^{X_s}(X_t \in B) \\
&= P_{0,t}(X_s, B).
\end{aligned}$$

Another way to obtain Markov processes is by solving certain stochastic differential equations depending on a starting point $x \in \mathbb{R}^d$ (see Section 2.6).
The connection between $\mathbb{P}^x$ and $\mathbb{P}^\mu$ is given by:

$$\mathbb{P}^\mu(G) = \int_{\mathbb{R}^d} \mathbb{P}^x(G) \, d\mu(x) \quad \text{for every } G \in \mathcal{G}.$$

Therefore, the Markov property is equivalent to

$$\mathbb{P}^\mu(X_{s+t} \in B \mid \mathcal{F}^0_s) = \mathbb{P}^{X_s}(X_t \in B) \quad \mathbb{P}^\mu\text{-a.s.}$$

for every $\mu \in M^1(\mathbb{R}^d), B \in \mathcal{B}^d$.
If X is the canonical process (on $(\mathbb{R}^d)^{\mathbb{R}_+}$ or $D(\mathbb{R}_+, \mathbb{R}^d)$) it is possible to introduce so called **shift operators** $\vartheta_t : \Omega \longrightarrow \Omega$. This is a semigroup of operators $(\vartheta_t)_{t \geq 0}$ such that: $X_s(\vartheta_t(\omega)) = X_{t+s}(\omega)$ for every $\omega \in \Omega,\ t, s \geq 0$. The effect of these is to cut off the part of the path before the time t and to shift the remaining part backwards in time.[2] Using these operators we are able to give a handy form of the Markov property: For every $t \geq 0$ and every Z which is $\mathcal{F}^0_\infty$-measurable and positive (or bounded):

$$\mathbb{E}^\mu(Z \circ \vartheta_t \mid \mathcal{F}^0_t) = \mathbb{E}^{X_t}(Z) \quad \mathbb{P}^\mu\text{-a.s.}$$

The formula says that the right-hand side, which is the composition of the two measurable maps $\omega \longmapsto X_t(\omega)$ and $x \longmapsto \mathbb{E}^x(Z)$ is in the equivalence class of the left-hand side.
Sometimes it is useful or even necessary to have the usual hypothesis (see (0.13)) satisfied for our processes. Since we are dealing later with Markov processes which are semimartingales, it is a natural assumption that there exists a version which has right

[1] Let us remark that the choice of $y \in \mathbb{R}^d$ is irrelevant since we are dealing with Markov processes, i.e. the future depends only on the position at time s and not on the starting point. However, since we have no $\mathbb{P}$ yet, we have to choose a starting point (and hence a $\mathbb{P}^x$) and the calculation shows that (a posteriori) there is no dependence on y.

[2] One can define these operators on non-canonical spaces as well, but one looses the intuitive meaning.

continuous paths. For Feller processes this is always the case. Generalizations beyond this class become immediately very technical (see e.g. [57] Section I.2).

From now on we assume that all Markov processes we encounter are right-continuous.

Having denoted the natural filtration by $\mathcal{F}_t^0 = \sigma(X_s : s \leq t)$ and $\mathcal{F}_\infty^0 = \vee_t \mathcal{F}_t^0$ we now switch to the completion: We denote by $\mathcal{F}_\infty^\mu$ the μ-completion of $\mathcal{F}_\infty^0$ and let

$$\mathcal{F}_t^\mu := \sigma(\mathcal{F}_t^0, \mu - \text{nullsets of } \mathcal{F}_\infty^\mu).$$

Finally we set

$$\mathcal{F}_\infty = \cap_\mu \mathcal{F}_\infty^\mu \qquad \text{and} \qquad \mathcal{F}_t = \cap_\mu \mathcal{F}_t^\mu$$

where the intersection is taken over all probability-measures on $(\Omega, \mathcal{F}_\infty^0)$. Since the process is right-continuous the completed filtration is automatically right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t} \mathcal{F}_s$ (see [52] Section III.2) and the process

$$\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, (\vartheta_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$$

is still Markovian.

A short comment on **killed processes** (see [52] page 79):
We assumed so far that $P_t(x, \mathbb{R}^d) = 1$ for every $t \geq 0$; a process with this property is called **conservative**. However, there are interesting processes with $P_t(x, \mathbb{R}^d) < 1$. In this case we say that the transition function is sub-Markovian. The intuition is that the process (or a modeled particle) is killed at a certain random time. We will denote this time in the sequel by ζ.
There is a nice technique which allows us to turn the sub-Markovian case into the Markovian one studied above. To this end we introduce a new point $\Delta \notin \mathbb{R}^d$, which we call the **cemetery**, and we set $\mathbb{R}^d_\Delta := \mathbb{R}^d \cup \{\Delta\}$ and $\mathcal{B}^d_\Delta = \sigma(\mathcal{B}^d, \{\Delta\})$. Furthermore, we define the transition function $\tilde{P}$ on $(\mathbb{R}^d_\Delta, \mathcal{B}^d_\Delta)$ by

$$\begin{aligned}
\tilde{P}_t(x, A) &= P_t(x, A) \qquad && \text{if } x \in \mathbb{R}^d,\ A \in \mathcal{B}^d \\
\tilde{P}_t(x, \{\Delta\}) &= 1 - P_t(x, \mathbb{R}^d) && \\
\tilde{P}_t(\Delta, \{\Delta\}) &= 1. &&
\end{aligned}$$

In the sequel, we will not distinguish between P and $\tilde{P}$. Consequently, all functions f on $\mathbb{R}^d$ will be extended to $\mathbb{R}^d_\Delta$ by setting $f(\Delta) = 0$ and the Markov property then reads

$$\mathbb{E}^\nu(Z \circ \vartheta_t | \mathcal{F}_t) = \mathbb{E}^{X_t}(Z) \ \ \mathbb{P}^\nu\text{-a.s. on the set } \{X_t \neq \Delta\},$$

because our convention implies that the right-hand side vanishes on $\{X_t = \Delta\}$, while the left-hand side does not vanish.

In the following chapters we will be most of the time in the conservative setting. However, sometimes it is useful to have some more flexibility.

On some occasions the usual Markov property is not sufficient. Before introducing the strong Markov property we first generalize the notion of shift operators: let T be a stopping time. $\vartheta_T : \Omega \longrightarrow \Omega$ is defined by:

$$\vartheta_T(\omega) := \begin{cases} \vartheta_t(\omega) & \text{if} \quad T(\omega) = t < +\infty \\ \omega_\Delta & \text{if} \quad T(\omega) = +\infty \end{cases}$$

where ω_Δ is the path identically equal to Δ. The **strong Markov property** reads with this convention: if T is a stopping time and Z is $\mathcal{F}_\infty$-measurable and positive (or bounded) then

$$\mathbb{E}^\mu(Z \circ \vartheta_T \mid \mathcal{F}_T) = \mathbb{E}^{X_T}(Z) \quad \mathbb{P}^\mu\text{-a.s}$$

for every $\mu \in M^1(\mathbb{R}^d)$ on the set $\{X_T \neq \Delta\}$.

Definition 1.5 If X is a strong Markov process which is quasi-left-continuous (see (0.25)), we call it a **Hunt process**.

On some occasions we will refer to the paper [13] by Cinlar, Jacod, Protter and Sharpe. Therefore we give a few remarks on how our thesis fits into their setting: first of all our processes are supposed to be normal, i.e $\mathbb{P}^x(X_0 = x) = 1$. This is one of the standard assumptions of the paper. In [13] the authors describe three basic setups under which the theorems are proved. We are always in the setup of their convention (3.6)(i). This means in detail:

In [13]:	Here:
$\mathcal{E}_0$	$(\mathcal{B}^d)^*$
$\mathcal{E}^*$	$(\mathcal{B}^d)^*$
$\mathcal{H}_t$	$\mathcal{F}_{t+} = \mathcal{F}_t$
$\mathcal{H}$	$\mathcal{F}_\infty$

And in (3.23) we have $\mathcal{H}'_t = \mathcal{F}_t$. Under this hypothesis their generalized strong Markov property becomes the usual strong Markov property.

1.3 Semigroups and Different Kinds of Generators

We associate a family of operators on $B_b(\mathbb{R}^d)$ with the (universal) Markov process X:

$$T_t u(x) := \mathbb{E}^x(u(X_t)) \qquad \text{for } t \geq 0 \text{ and } u \in B_b(\mathbb{R}^d). \tag{1.4}$$

Recall that the space $B_b(\mathbb{R}^d)$ equipped with the sup-norm $\|\cdot\|_\infty$ is a Banach space.

Theorem 1.6 *The family $(T_t)_{t\geq 0}$ of operators is a positivity preserving contraction semigroup on $B_b(\mathbb{R}^d)$. This means in detail:*

$$\begin{array}{lll} (a) & T_t : B_b(\mathbb{R}^d) \longrightarrow B_b(\mathbb{R}^d) & \text{linear operator on } B_b(\mathbb{R}^d) \\ (b) & \|T_t u\|_\infty \leq \|u\|_\infty & \text{contraction} \\ (c) & T_0 = \mathrm{id} & \\ (d) & T_{s+t} = T_s \circ T_t & \text{semigroup} \\ (e) & 0 \leq u \leq 1 \Rightarrow 0 \leq T_t u \leq 1 & \text{positive, sub-Markovian.} \end{array}$$

If the process is conservative (i.e. $P_t(x, \mathbb{R}^d) = 1$ for all $t \geq 0, x \in \mathbb{R}^d$), we have in addition:

$$(f) \quad T_t 1 = 1 \qquad\qquad \text{Markovian.}$$

Example 1.7 If Z is a Lévy process (see Section 1.4), then the semigroup $(T_t)_{t\geq 0}$ is given by

$$T_t u(x) = \int_{\mathbb{R}^d} u(x+y)\ \mu_t(dy)$$

where $(\mu_t)_{t\geq 0}$ is the vaguely continuous convolution semigroup of measures which is associated with Z (See [5] page 317).

Following [15] we denote by L_0 the set of all functions $u \in B_b(\mathbb{R}^d)$ for which

$$(g) \quad \lim_{t\downarrow 0} \|T_t u - u\|_\infty = 0 \qquad\qquad \text{strongly continuous.}$$

Let us remark that L_0 is a closed subspace of $B_b(\mathbb{R}^d)$.

An important concept in the analysis of a strongly continuous contraction semigroup is that of the generator. This is a mapping which is defined on a subspace $D(A)$ of $B_b(\mathbb{R}^d)$.

Definition 1.8 We define the **generator** of the semigroup $(T_t)_{t\geq 0}$ to be the linear mapping $A : D(A) \longrightarrow B_b(\mathbb{R}^d)$:

$$Au := \lim_{t\downarrow 0} \frac{T_t u - u}{t} \qquad , u \in D(A)$$

where

$$D(A) := \left\{ u \in B_b(\mathbb{R}^d) : \lim_{t\downarrow 0} \frac{T_t u - u}{t} \text{ exists in } \|\cdot\|_\infty \right\}$$

is the **domain** of the operator.

Proposition 1.9 *a) The semigroup $(T_t)_{t\geq 0}$ leaves the subspace L_0 invariant.*

b) For every $u \in D(A)$ we have $Au \in L_0$.

c) $D(A) \subset L_0$ and L_0 is the $\|\cdot\|_\infty$-closure of $D(A)$.

d) If $u \in D(A)$ then

$$T_t u - u = \int_0^t T_s A u\ ds.$$

e) The operator A is closed.

Proof: See [15] Section I.2. □

Example 1.10 One-dimensional Brownian motion is the Markov process which is associated with the semigroup on $B_b(\mathbb{R}^d)$

$$T_t u(x) = \int_{\mathbb{R}} u(x+y) \cdot \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}\, dy.$$

The generator of Brownian motion is $-\Delta$, where Δ denotes the Laplacian operator. We do not give a proof for this result here, because it directly follows from the more general case of Lévy processes.
It is a natural question to ask what the space L_0 looks like: let $u \in B_b(\mathbb{R}^d)$. It follows by a linear transformation and Lebesgue's theorem that

$$T_t u(x_n) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u(z) e^{-(z-x_n)^2/2t} dz \xrightarrow[x_n \to x]{} T_t u(x),$$

and

$$T_t u(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u(z) e^{-(z-x)^2/2t} dz \xrightarrow[x \to \infty]{} 0,$$

i.e. $T_t : B_b(\mathbb{R}^d) \longrightarrow C_\infty(\mathbb{R}^d)$. But for $u \in L_0$ we have that $T_t u$ tends uniformly to u as t goes to zero. Therefore, we conclude that $u \in C_\infty(\mathbb{R}^d)$. On the other hand if $u \in C_\infty(\mathbb{R}^d)$ it is clear that

$$T_t u(x) \xrightarrow[t \downarrow 0]{} u(x).$$

Putting this together we obtain $L_0 = C_\infty(\mathbb{R}^d)$ in the case of Brownian motion. Since we always want to go beyond this kind of processes, it makes sense to restrict the space in consideration to $C_\infty(\mathbb{R}^d)$.[3] This is what we will do in the next section.

That a (pseudo) differential operator appears in this setting is not a singular case, but a general phenomenon. We first give the definition for this kind of operator. Recall that $\widehat{u}$ denotes the Fourier transform.

Definition 1.11 An operator $p(x, D)$ on the Schwartz space $\mathrm{S}(\mathbb{R}^d)$ is called **pseudo differential operator** if it can be written as

$$p(x, D)u(x) = \int_{\mathbb{R}^d} e^{ix'\xi} p(x, \xi) \widehat{u}(\xi)\, d\xi \qquad \text{for } u \in \mathrm{S}(\mathbb{R}^d)$$

where $p : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ has the following properties:

- It is locally bounded in x, ξ.
- $p(\cdot, \xi)$ is measurable for every ξ.
- $p(x, \cdot)$ is a continuous negative definite functions for every x.

In this case we call p the **(continuous negative definite) symbol** of the operator.

[3] In fact one does not have to use Brownian motion in this consideration. Every Lévy process with a C_∞-transition density would have served as well.

Remarks: a) The last point essentially means that $p(x,\cdot)$ has a Lévy-Khinchine representation, see Section 1.1:

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y\cdot\chi(y)\right) N(x,dy)$$

where every part of the **Lévy triplet** $(\ell(x), Q(x), N(x,dy))$ is now 'state space dependent'.

b) This definition makes sense, since the symbol is of maximal polynomial growth in the second argument (see Proposition 1.2 j) and $\widehat{u} \in \mathrm{S}(\mathbb{R}^d)$, therefore, the integral exists.

Proposition 1.12 *The generator A fulfills the so called **positive maximum principle** (PMP), i.e.*

$$u \in D(A),\ x_0 \in \mathbb{R}^d \text{ and } \sup_{x\in\mathbb{R}^d} u(x) = u(x_0) \geq 0 \ \Rightarrow\ Au(x_0) \leq 0. \tag{1.5}$$

Proof: Let $u \in D(A)$ and x_0 be such that $\sup_{x\in\mathbb{R}^d} u(x) = u(x_0) \geq 0$. By the assumption from above and properties (b) and (e) of Theorem 1.6 we have

$$u(x_0) \geq \left\|u^+\right\|_\infty \geq \left\|Tu^+\right\|_\infty \geq \|Tu\|_\infty \geq Tu(x_0)$$

which yields

$$\frac{Tu(x_0) - u(x_0)}{t} \leq 0$$

The assertion follows by passing to the limit. □

Linear mappings with this property which are defined on the test functions are well understood and there is a nice structural result which reads as follows:

Theorem 1.13 ***(Courrège)*** *If A is an operator which maps the test functions $C_c^\infty(\mathbb{R}^d)$ into the continuous functions and satisfies the positive maximum principle, then $-A$ is a pseudo differential operator.*

Proof: See [14] Theorem 3.4 or [33] Theorem 4.5.21 □

For our purposes we need the following corollary:

Corollary 1.14 *The generator A of a Markov semigroup with $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and $A(C_c^\infty(\mathbb{R}^d)) \subset C(\mathbb{R}^d)$ has a representation on the test functions as follows:*

$$\begin{aligned} A|_{C_c^\infty(\mathbb{R}^d)} u(x) &= -p(x,D)u(x) \\ &:= -\int_{\mathbb{R}^d} e^{ix'\xi} p(x,\xi)\widehat{u}(\xi)\ d\xi \end{aligned} \tag{1.6}$$

where $p(x,\xi)$ is a continuous negative definite symbol.

In the literature one finds different extensions of the generator. In the following we describe the interdependence between the various concepts:

Definition 1.15 The **full generator** $\hat{A}$ of $(T_t)_{t\geq 0}$ is defined to be:

$$\hat{A} := \left\{(u, w) \in B_b(\mathbb{R}^d) \times B_b(\mathbb{R}^d) = T_t u - u = \int_0^t T_s w \; ds, \; t \geq 0\right\}.$$

Remarks: a) In general $\hat{A}$ is not single valued.
b) Proposition 1.7 d) shows that if u is in $D(A)$ then $(u, Au) \in \hat{A}$. Equivalently one could write $D(A) \subset \pi_1(\hat{A})$ where π_1 is the projection of the set onto the first component

The following theorem gave rise to a variety of investigations:

Theorem 1.16 *Let $(X_t)_{t\geq 0}$ be an $\mathbb{F}^0$-progressive Markov process. $\hat{A}$ be the full generator of its semigroup $(T_t)_{t\geq 0}$. Then for every $(u, w) \in \hat{A}$*

$$C_t^{(u)} := u(X_t) - \int_0^t w(X_s) \; ds$$

is an $(\mathcal{F}^0, \mathbb{P}^x)$-martingale with respect to every $x \in \mathbb{R}^d$.

Proof: We follow closely [17] Proposition 2.1.7. The notion of stochastic integrals will be introduced in Section 2.3. However, the integrals below can be calculated as pathwise Lebesgue integrals. Fix $x \in \mathbb{R}^d$ and let $t, h \geq 0$:

$$\begin{aligned}
\mathbb{E}^x(C_{t+h}^{(u)} | \mathcal{F}_t^0) &= \mathbb{E}^x(u(X_{t+h})|\mathcal{F}_t^0) - \mathbb{E}^x\left(\textstyle\int_t^{t+h} w(X_s) \; ds \; |\mathcal{F}_t^0\right) - \mathbb{E}^x\left(\textstyle\int_0^t w(X_s) \; ds \; |\mathcal{F}_t^0\right) \\
&= \mathbb{E}^{X_t}\left(u(X_h)\right) - \mathbb{E}^{X_t}\left(\textstyle\int_0^h w(X_s) \; ds\right) - \textstyle\int_0^t w(X_s) \; ds \\
&= \mathbb{E}^y\left(u(X_h)\right)|_{y=X_t} - \mathbb{E}^y\left(\textstyle\int_0^h w(X_s) \; ds\right)|_{y=X_t} - \textstyle\int_0^t w(X_s) \; ds \\
&= T_h u(y)|_{y=X_t} - \textstyle\int_0^h T_s w(y) \; ds|_{y=X_t} - \textstyle\int_0^t w(X_s) \; ds \\
&= u(X_t) - \textstyle\int_0^t w(X_s) \; ds \\
&= C_t^{(u)}
\end{aligned}$$

where we used the Markov property (MP1) in the second equation and the fact that $\int_0^t w(X_s) \; ds$ is measurable with respect to $\mathcal{F}_t^0$. □

Definition 1.17 Let X be a Markov process. A function $u \in B_b(\mathbb{R}^d)$ is said to belong to the **domain of the extended generator** of X, written as $u \in D(A_{ext})$, if there exists a $w \in (\mathcal{B}^d)^*$ such that the process

$$M_t^{[u]} = u(X_t) - u(X_0) - \int_0^t w(X_s) \; ds$$

is well defined and a local martingale (see Section 2.1) for every $\mathbb{P}^x$. If we choose for every $u \in D(A_{ext})$ one w with this property, we write $A_{ext}u := w$ and call A_{ext} (a version of) the **extended generator** of X.

Remarks: a) The extended generator is well defined since it is unique up to a set $A \subset \mathbb{R}^d$ of potential zero, i.e.

$$\mathbb{P}^x\Big(\int_0^\infty 1_A(X_s)\, ds > 0\Big) = 0.$$

See [13] Remarks (7.3).
b) In the above proposition it was shown that if $u \in \pi_1(\hat{A})$ then there is a w such that M_t is a martingale with respect to every $\mathbb{P}^x$. Therefore, $M_t^{[u]} = C_t^{(u)} - u(X_0)$ is a local martingale. This means

$$D(A) \subset \pi_1(\hat{A}) \subset D(A_{ext}).$$

In our investigations of so called Itô processes the extended generator plays a fundamental role.

It is also important in our considerations if the process is conservative (i.e. has almost surely infinite lifetime). The following criterion is proved in [54]. It perfectly fits into our context since it uses the notion of the symbol.

Theorem 1.18 *Let $(X_t)_{t\geq 0}$ be a (universal) Markov process with càdlàg paths and generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and $A|C_c^\infty(\mathbb{R}^d) = -p(x, D)$ with symbol $p(x,\xi)$. Then $(X_t)_{t\geq 0}$ is conservative, if*

$$\lim_{k\to\infty} \sup_{\|y-x\|\leq 2k} \sup_{\|\eta\|\leq \frac{1}{k}} |p(y,\eta)| = 0$$

for all $x \in \mathbb{R}^d$.

1.4 Feller Processes, Feller Semigroups and Their Generators

As we mentioned in Section 1.2 it is possible to associate a semigroup of operators to X by setting for every $t \geq 0$

$$T_t u(x) := \mathbb{E}^x u(X_t) \qquad \text{for } u \in B_b(\mathbb{R}^d).$$

Some of its basic properties were stated above. If the semigroup fulfills two additional properties, we call the process X and the semigroup 'Feller':

Definition 1.19 If the semigroup $(T_t)_{t\geq 0}$ satisfies

(g)	$\lim_{t\downarrow 0} \|T_t u - u\|_\infty = 0$ for all $u \in C_\infty(\mathbb{R}^d)$	strongly continuous
(h)	$T_t : C_\infty(\mathbb{R}^d) \longrightarrow C_\infty(\mathbb{R}^d)$	operator on $C_\infty(\mathbb{R}^d)$

we call it a **Feller semigroup** or Fellerian. The associated process is then called a **Feller process**.

Let us remark that for every Feller process there exists a càdlàg version, i.e. it is possible to restrict the measure $\mathbb{P}$ on the path-space to the Skorokhod space $D(\mathbb{R}_+, \mathbb{R}^d)$. Without further mentioning we will always assume the Feller processes we encounter to be càdlàg (see e.g. [52] Theorem III.(2.7)).

Since $(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ is again a Banach space we have tools like the Hille-Yosida-Ray theorem (see [17] Theorem 1.2.6).

Definition 1.20 We will call a Feller process X with generator (A,D(A)) **nice**, if $C_c^\infty(\mathbb{R}^d) \subset D(A)$.

Corollary 1.14 reads in the present setting:

Corollary 1.21 *The generator A of a nice Feller process has a representation on the test functions as follows:*

$$\begin{aligned} A|_{C_c^\infty(\mathbb{R}^d)}u(x) &= -p(x,D)u(x) \\ &:= -\int_{\mathbb{R}^d} e^{ix'\xi}p(x,\xi)\widehat{u}(\xi)\ d\xi \end{aligned} \tag{1.7}$$

where $p(x,\xi)$ is a negative definite symbol with the same properties as in Corollary 1.14.

In anticipation of the next section we give the following example:

Example 1.22 Every Lévy process (with characteristic exponent ψ) is a nice Feller process and its generator - restricted to the test functions - is

$$A|_{C_c^\infty(\mathbb{R}^d)}u(x) = -\int_{\mathbb{R}^d} e^{ix'\xi}\psi(\xi)\widehat{u}(\xi)\ d\xi,\ u \in C_c^\infty(\mathbb{R}^d)$$

(see the following section).

Corollary 1.21 essentially says that we know something about the structure of the generator as soon as we know that the test functions are contained in the domain. There have been many investigations on the question for which negative definite symbols $p(x,\xi)$ the operator $-p(x,D)$ gives rise to the generator of a Feller process. For a survey on the different construction methods see [32] and also [34] and [35]. So called 'stable-like processes' are considered in [4], [30], [24] and [48]. Constructions making use of the martingale problem can be found in [23], [25] and [58]. For constructions via analytic methods see [29], [28], [10], [3] and [1]. In Section 5.1 this problem is solved for a particular class of symbols.

Theorem 1.16 reads in the present setting:

Theorem 1.23 *Let $(X_t)_{t\geq 0}$ be an $\mathbb{F}$-adapted Feller process. Let A be the generator of its semigroup $(T_t)_{t\geq 0}$. Then for every $u \in D(A)$*

$$M_t^{[u]} := u(X_t) - u(X_0) - \int_0^t Au(X_s)\ ds$$

is an $(\mathbb{F}, \mathbb{P}^\mu)$-martingale with respect to every initial-measure $\mu \in M^1(\mathbb{R}^d)$.

Proof: See [52] Proposition VII.(1.6).

Of course Theorem 1.15 works in this context. Another result on the conservativeness of Feller processes can be found in [25] Chapter 9.

1.5 Lévy Processes

Starting from a convolution semigroup we obtain a special subclass of Feller processes. **Convolution semigroups** are families of probability measures on $\mathcal{B}^d$: $(\mu_t)_{t\geq 0}$ such that

$$\begin{array}{lll} (CSG1) & \mu_t(\mathbb{R}^d) = 1 & \text{for every } t \geq 0 \\ (CSG2) & \mu_s * \mu_t = \mu_{s+t} & \text{for every } s,t \geq 0 \text{ and } \mu_0 = \delta_0 \\ (CSG3) & \mu_t \to \delta_0 & \text{vaguely as } t \to 0 \end{array}$$

The last point means that for every $u \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} u(y)\ \mu_t(dy) \underset{t\downarrow 0}{\longrightarrow} \int_{\mathbb{R}^d} u(y)\ \delta_0(dy) = u(0).$$

Since we are dealing with probability measures, this is equivalent to weak convergence (testing against $C_b(\mathbb{R}^d)$-functions). Let us also mention that there is a one-to-one correspondence between convolution semigroups $(\mu_t)_{t\geq 0}$ on $\mathcal{B}^d$ and continuous negative definite functions $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ given by

$$\hat{\mu}_t(\xi) := \int_{\mathbb{R}^d} e^{i\xi' y}\ \mu_t(dy) = e^{-t\psi(\xi)} \text{ for every } t \geq 0,\ \xi \in \mathbb{R}^d.$$

(See [33] Theorem 3.6.16.)

It is easy to see that by setting

$$P_t(z,B) := \int_{\mathbb{R}^d} 1_B(z+y)\ \mu_t(dy) \text{ for } t \geq 0,\ z \in \mathbb{R}^d,\ B \in \mathcal{B}^d$$

we obtain a transition semigroup $(P_t)_{t\geq 0}$, which is translation invariant in space ($h \in \mathbb{R}^d$):

$$P_t(z+h,B) = \int_{\mathbb{R}^d} 1_B(z+h+y)\ \mu_t(dy) = P_t(z, B-h).$$

By adding an initial measure $\mu \in M^1(\mathcal{B}^d)$ we obtain by the formula

$$\begin{aligned} &\mathbb{P}_{t_1,\dots,t_n}(B_0, B_1, \dots, B_n) \\ &\quad := \int_{B_0}\int_{B_1} \dots \int_{B_n} P_{t_n - t_{n-1}}(z_{n-1}, dz_n) \dots P_{t_1}(z, dz_1)\ \mu(dz) \end{aligned} \tag{1.8}$$

of Section 1.2 the finite dimensional distributions of a (simple) Markov process $(Z_t)_{t\geq 0}$.

A process of this kind is called **Lévy process**. It has the following properties (with respect to $\mathbb{P}^\mu$):

$(L0)$	$\text{law}(Z_0) = \mu$	initial distribution
$(L1)$	$\text{law}(Z_{s+h} - Z_s) = \text{law}(Z_{t+h} - Z_t)$ for $s, t, h \geq 0$	stationary increments
$(L2)$	For $n \in \mathbb{N}$, $t_0 < t_1 < ... < t_n$ the following r.v.'s are independent: $Z_{t_0}, (Z_{t_1} - Z_{t_0}), ..., (Z_{t_n} - Z_{t_{n-1}})$	indep. increments
$(L3)$	$\lim_{t \to s} \mathbb{P}(\|Z_t - Z_s\| > \varepsilon) = 0$ for all $s \geq 0, \varepsilon > 0$	stoch. continuous

Conversely there exists for every process $(Z_t)_{t \geq 0}$ with these four properties a convolution semigroup generating the process. The relationship between these objects is given by:

$$\text{law}(Z_t - Z_s) = \mu_{t-s} \text{ and } \mu = \mu_0$$

under each measure $\mathbb{P}^\mu$.

From the point of view of universal Markov processes it is more interesting to investigate the whole family

$$(\Omega, \mathcal{F}, \mathbb{F}, (Z_t)_{t \geq 0}, \mathbb{P}^z)_{z \in \mathbb{R}^d}.$$

Since the transition function is translation invariant we obtain

$$\begin{aligned} \mathbb{E}^z 1_B(Z_t) &= \mathbb{P}^z(Z_t \in B) = P_t(z, B) = P_t(0, B - z) \\ &= \mathbb{P}^0(Z_t \in B - z) = \mathbb{E}^0 1_{B-z}(Z_t) = \mathbb{E}^0 1_B(Z_t + z) \end{aligned} \tag{1.9}$$

and by passing from step functions to positive measurable functions and then to $u \in B_b$:

$$\begin{aligned} T_t u(z) &= \mathbb{E}^z u(Z_t) = \mathbb{E}^0 u(Z_t + z) = \int_\Omega u(Z_t + z) \, d\mathbb{P}^0 \\ &= \int_{\mathbb{R}^d} u(y + z) \, \mathbb{P}^0(Z_t \in dy) = \int_{\mathbb{R}^d} u(z + y) \, \mu_t(dy). \end{aligned}$$

It turns out that such a semigroup (and hence the process) is Fellerian: Let $u \in C_\infty$. Fix $t \geq 0$ and consider a sequence $(z_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$ such that $z_n \to z \in \mathbb{R}^d$. We have $u(z_n + y) \to u(z + y)$ for every $y \in \mathbb{R}^d$, because u is continuous. By Lebesgue's theorem we obtain

$$\int_{\mathbb{R}^d} u(z_n + y) \, \mu_t(dy) \xrightarrow[z_n \to z]{} \int_{\mathbb{R}^d} u(z + y) \, \mu_t(dy).$$

Analogously it is shown, that $T_t u(z)$ vanishes at infinity, hence (h) of Definition 1.19. Since $(\mu_t)_{t \geq 0}$ is weakly continuous and $y \longmapsto u(z + y)$ is in C_∞ for every $u \in C_\infty$ and $z \in \mathbb{R}^d$ we obtain

$$T_t u(z) = \int_{\mathbb{R}^d} u(z + y) \, \mu_t(dy) \xrightarrow[t \downarrow 0]{} \int_{\mathbb{R}^d} u(z + y) \, \mu_0(dy) = T_0 u(z).$$

It is enough to show pointwise convergence (see [52] Proposition III.(2.4)), since we have already proved (h). As every Lévy processes is a Feller processes, we can and will assume that it is càdlàg.

Let us remark that some authors require that a Lévy processes starts a.s. in zero. Setting $\mu := \delta_0$ this case is included in our considerations. Conversely one could begin with a Lévy processes starting a.s. in zero and use formulas (1.8) and (1.9) in order to obtain the universal Markov process from above (compare Section 40 of [53]).

Sometimes it is useful to see the universal Markov process as a family of processes $(Z^z)_{z\in\mathbb{R}^d}$, where z denotes the starting point of each process, and one probability measure $\mathbb{P}$. This is expressed by

$$\mathbb{P}^z_{Z_t}(dy) = \mathbb{P}^z(Z_t \in dy) = \mathbb{P}(Z_t^z \in dy) = \mathbb{P}_{Z_t^z}(dy).$$

On the left-hand side we have one process and a family of measures on the right-hand side there is only one measure, but a family of processes. One side given it is always possible to write the other one on the canonical space.

The following theorems will be very useful in all that follows:

Theorem 1.24 *If $(Z_t)_{t\geq 0}$ is a (conservative) Lévy process with $Z_0 = 0$, then it has the following **Lévy-Itô decomposition**:*

$$\begin{aligned} Z_t \;=\; & \underbrace{\sigma W_t}_{\text{Gaussian part}} + \underbrace{\int_{[0,t]\times\{\|y\|<1\}} y\,\Big(\mu^Z(ds,dy) - ds N(dy)\Big)}_{\text{compensated small jumps}} && L^2\text{-martingale} \\ & + \underbrace{\ell t}_{\text{drift}} + \underbrace{\sum_{0<s\leq t}\Big(\Delta Z_s 1_{\{\|\Delta Z_s\|\geq 1\}}\Big)}_{\text{big jumps}} && \text{finite variation} \end{aligned}$$

*where the four terms are independent and ℓ is a vector in $\mathbb{R}^d$, σ is a positive semidefinite $d\times d$-matrix and W a standard Brownian motion, μ^Z is the random point measure associated to the jumps of Z (see Section 2.2) and N is a measure on $\mathbb{R}^d\backslash\{0\}$ such that $\int(1\wedge y^2)\;N(dy) < \infty$, the so called **Lévy measure**.*

Proof: See [53] Section 20. □

Let us remark that this representation corresponds to the cut-off function $1_{\{\|y\|<1\}}$. With another χ we still would obtain a Lévy-Itô decomposition. Closely linked to this decomposition is the **Lévy-Khinchine formula**.

Theorem 1.25 *If $Z = (Z_t)_{t\geq 0}$ is a (conservative) Lévy process, then the characteristic function of Z_t can be written as*

$$\mathbb{E}^z\left(e^{i\xi'(Z_t - z)}\right) = \mathbb{E}^0\left(e^{i\xi' Z_t}\right) = e^{-t\cdot\psi(\xi)} \tag{1.10}$$

where

$$\psi(\xi) = -i\ell'\xi + \frac{1}{2}\xi' Q\xi - \int_{y\neq 0}\left(e^{i\xi' y} - 1 - i\xi' y\cdot\chi(y)\right)\;N(dy) \tag{1.11}$$

with $\ell \in \mathbb{R}^d$, $Q = \sigma^2$ is a positive semidefinite matrix, N is the Lévy measure from above and χ is a cut-off function (see Definition 1.4).
Conversely there exists for every such ψ a (unique) Lévy process Z, such that (1.10) *holds. The function ψ is called* ***characteristic exponent*** *of the Lévy process Z.*

Proof: See [53] Section 8. Note that Lévy processes are in this thesis always assumed to be càdlàg. □

Proposition 1.26 *Let Z be a d-dimensional stochastic process.*
a) If Z is a Lévy process, then the component processes $Z^{(j)}$ $(j = 1, ..., d)$ are Lévy processes.
b) If the $Z^{(j)}$ $(j = 1, ..., d)$ are independent Lévy processes, then Z is a Lévy process.

Proof: a) Let Z be a Lévy process and (w.l.o.g.) $j = 1$. We have

$$\mathbb{P}(Z_t^{(1)} - Z_s^{(1)} \in B) = \mathbb{P}(Z_t - Z_s \in B \times \mathbb{R}^{d-1}) \text{ for every } B \in \mathcal{B}^1.$$

Hence (L1) and (L3) of $Z^{(1)}$ follow directly from the respective properties of Z.
A similar argumentation yields (L2): For $A, B \in \mathcal{B}^1$ and $s \le t \le u \le v$ we have

$$\begin{aligned} &\mathbb{P}(Z_t^{(1)} - Z_s^{(1)} \in A, Z_v^{(1)} - Z_u^{(1)} \in B) \\ &\quad= \mathbb{P}(Z_t - Z_s \in A \times \mathbb{R}^{d-1}, Z_v - Z_u \in B \times \mathbb{R}^{d-1}) \\ &\quad= \mathbb{P}(Z_t - Z_s \in A \times \mathbb{R}^{d-1}) \cdot \mathbb{P}(Z_v - Z_u \in B \times \mathbb{R}^{d-1}) \\ &\quad= \mathbb{P}(Z_t^{(1)} - Z_s^{(1)} \in A) \cdot \mathbb{P}(Z_v^{(1)} - Z_u^{(1)} \in B). \end{aligned}$$

b) This works analogously to a). Let us just remark that for $B_1, ..., B_d \in \mathcal{B}^1$ we have

$$\mathbb{P}(Z_t - Z_s \in B_1 \times ... \times B_d) = \mathbb{P}(Z_t^{(1)} - Z_t^{(1)} \in B_1) \cdot ... \cdot \mathbb{P}(Z_t^{(d)} - Z_t^{(d)} \in B_d)$$

and that the rectangles $B_1 \times ... \times B_d$ generate the product σ-algebra $\mathcal{B}^d$. □

Remark: Let ψ_j be the characteristic exponent of the j-th component of the process Z in Proposition 1.26 b). We obtain for the characteristic function, using the independence,

$$\begin{aligned} \mathbb{E}^0\left(e^{i\xi' Z_t}\right) &= \mathbb{E}^0\left(\exp(i\xi^{(1)} Z_t^{(1)} + ... + \xi^{(d)} Z_t^{(d)})\right) \\ &= \mathbb{E}^0\left(\exp(i\xi^{(1)} Z_t^{(1)})\right) \cdot ... \cdot \mathbb{E}^0\left(\exp(i\xi^{(d)} Z_t^{(d)})\right) \\ &= \mathbb{E}^0\left(e^{-t(\psi_1(\xi^{(1)}) + ... + \psi_d(\xi^{(d)}))}\right). \end{aligned}$$

On the level of characteristic exponents this reads:

$$\psi(\xi) = \psi_1(\xi^{(1)}) + ... + \psi_d(\xi^{(d)}).$$

This result will be generalized in Section 4.4.

Next we calculate the generator of a Lévy process.

Theorem 1.27 *On the Schwartz space the generator A of a Lévy process can be written in the following way*

$$Au(z) = \int_{\mathbb{R}^d} \widehat{u}(\xi)(-\psi(\xi)) \cdot e^{iz'\xi} \, d\xi \tag{1.12}$$

for $u \in \mathrm{S}(\mathbb{R}^d)$, i.e. $-A$ is a pseudo differential operator and the symbol ψ is the characteristic exponent.

Proof: Let u be a function in the Schwartz space $\mathrm{S}(\mathbb{R}^d)$ and consider

$$
\begin{aligned}
&\left| \frac{T_t u(z) - u(z)}{t} - Au(z) \right| \\
&\quad = \left| \frac{1}{t} \Big(\mathbb{E}^z(u(Z_t)) - u(z) \Big) - Au(z) \right| \\
&\quad = \left| \frac{1}{t} \left(\mathbb{E}^z \int_{\mathbb{R}^d} \big(\widehat{u}(\xi) e^{iy'\xi}|_{y=Z_t(\omega)} \big) \, d\xi - u(z) \right) - Au(z) \right| \\
&\quad = \left| \frac{1}{t} \left(\int_{\mathbb{R}^d} \big(\widehat{u}(\xi) \mathbb{E}^z (e^{iZ_t(\omega)'\xi}) \big) \, d\xi - u(z) \right) - Au(z) \right| && \text{Fubini} \\
&\quad = \left| \frac{1}{t} \left(\int_{\mathbb{R}^d} \big(\widehat{u}(\xi) (e^{-t\psi(\xi)}) \cdot e^{iz'\xi} \big) \, d\xi - u(z) \right) - Au(z) \right| && \text{LKF} \\
&\quad = \left| \int_{\mathbb{R}^d} \left(\widehat{u}(\xi) \frac{e^{-t\psi(\xi)} - 1}{t} \cdot e^{iz'\xi} \right) d\xi - \int_{\mathbb{R}^d} \left(\widehat{u}(\xi)(-\psi(\xi)) \cdot e^{iz'\xi} \right) d\xi \right| \\
&\quad = \left| \int_{\mathbb{R}^d} \left(\widehat{u}(\xi) \frac{e^{-t\psi(\xi)} + t\psi(\xi) - 1}{t} \cdot e^{iz'\xi} \right) d\xi \right| \\
&\quad \le \int_{\mathbb{R}^d} \left(|\widehat{u}(\xi)| \left| \frac{e^{-t\psi(\xi)} + t\psi(\xi) - 1}{t} \right| \right) d\xi \\
&\quad \le \frac{1}{2} \int_{\mathbb{R}^d} \left(|\widehat{u}(\xi)| \, \frac{1}{t} \cdot t^2 \sup_{\eta \in \mathbb{R}^d, \|\eta\| \le \|\xi\|} |\psi(\eta)|^2 \right) d\xi && \text{Taylor} \\
&\quad \le t \cdot \frac{1}{2} \cdot \int_{\mathbb{R}^d} \left(|\widehat{u}(\xi)| \, c_\psi (1 + \|\xi\|^2) \right) d\xi. && \text{Prop. 1.2. j)}
\end{aligned}
$$

The last integral exists, because $\widehat{u}$ is in the Schwartz space. We obtain

$$t \cdot \frac{1}{2} \cdot \int_{\mathbb{R}^d} \left(|\widehat{u}(\xi)| \, c_\psi (1 + \|\xi\|^2) \right) d\xi \xrightarrow[t \downarrow 0]{} 0 \text{ uniformly in } z,$$

as the integral is finite and does not depend on t or z. □

Since the test functions are contained in the domain of the generator this means that a Lévy process is the spatially homogeneous case of (nice) Feller processes and the symbol of the process is just its characteristic exponent ψ, which is in this case independent of z.

Remarks: a) In the case of Lévy processes the test functions $C_c^\infty(\mathbb{R}^d)$ are a core of A, i.e. the smallest closed extension[4] of $A|_{C_c^\infty(\mathbb{R}^d)}$ equals A (see [53] Theorem 31.5).

b) We have $C_\infty^2(\mathbb{R}^d) \subset D(A)$ (ibid.).

c) Pointwise convergence would have been sufficient in the proof above (see [53] Lemma 31.7).

In this thesis we only deal with conservative Lévy processes. If we took a so called 'killing' into account the measures in (CSG1) $(\mu_t)_{t\geq 0}$ would be sub-probability measures. In the Lévy-Khinchine formula we would have an additional $a > 0$:

$$\psi(\xi) = a - i\ell'\xi + \frac{1}{2}\xi' Q\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y) \right) \, N(dy).$$

[4]Extension is meant in the sense of linear operators: the operator $(A', D(A'))$ is an extension of $(A, D(A))$ if $D(A) \subset D(A')$ and $A'|_{D(A)} = A$.

2 Semimartingales and Stochastic Integration

In this chapter we give a brief introduction to the theory of semimartingales. In particular we introduce the notion of the characteristic triplet. This turns out to be very important in the context of the stochastic symbol which is introduced in Chapter 4. The concepts related to stochastic integration are treated in Sections three to five. In the detailed Section 2.6 we are dealing with stochastic differential equations.

All the definitions in this chapter are meant either with respect to a single probability measure $\mathbb{P}$ or to a family of measures $(\mathbb{P}^x)_{x\in\mathbb{R}^d}$ as we encountered in the previous chapter. If not stated otherwise we fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Martingales and processes of finite variation are introduced one-dimensional. A d-dimensional semimartingale will be defined as a vector of semimartingales (cf. [50]).

2.1 Finite Variation and Localization

Processes of finite variation and local martingales are the main ingredients when dealing with semimartingales. Therefore, we introduce these concepts in the present section. Our standard reference for functions of finite variation is [47].

Definition 2.1 (i) If g is a real valued function on the interval $[a, b]$ then

$$V(g; [a, b]) := \sup_{\pi_n} \sum_{j=1}^{n} |g(t_j) - g(t_{j-1})| ,$$

where the supremum is taken over all partitions
$\pi_n = (a = t_0 < t_1 < ... < t_n = b)$ is called the (total) **variation** of g over $[a, b]$.
(ii) If g is defined on $\mathbb{R}_+$, then we set

$$V(g; t) := V(g; [0, t]).$$

(iii) We say that g is of **finite variation** (on compacts) if $V(g; t) < \infty$ for all $t \geq 0$.

Theorem 2.2 *Any function $g(t) : [0, \infty[\longrightarrow \mathbb{R}$ of finite variation can be expressed as the difference of two increasing functions*

$$g(t) = a(t) - b(t).$$

If g is right-continuous, then a and b can be chosen to be right-continuous.

Definition 2.3 An $\mathbb{R}$-valued stochastic process A is said to be of **finite variation**, if almost all paths of A are of finite variation on compacts (as functions). We use FV as an abbreviation.

Definition 2.4 (i) Let A be an adapted càdlàg process such that $A_0 = 0$.
The process is said to belong to class $\mathcal{V}^+$ (respectively to class $\mathcal{V}$) if it is increasing

(respectively of finite variation).

(ii) For an $A \in \mathcal{V}^+$ we define A_∞ as the following $]-\infty, \infty]$-valued random variable:

$$A_\infty := \lim_{t\to\infty} A_t.$$

(iii) We write $\mathrm{Var}(A)$ for the **variation process** of $A \in \mathcal{V}$, which is defined by $\mathrm{Var}(A)_t(\omega) := V(A_\bullet(\omega); t)$ for $t \geq 0$.

(iv) A process is said to belong to class $\mathcal{A}^+$ (respectively $\mathcal{A}$) if it is of class $\mathcal{V}^+$ and $\mathbb{E}(A_\infty) < \infty$ (respectively if it is of class $\mathcal{V}$ and $\mathbb{E}(\mathrm{Var}(A)_\infty) < \infty$).

Remarks: a) $\mathcal{V}^+ \subset \mathcal{V}$ and $\mathcal{V} = \mathcal{V}^+ - \mathcal{V}^+$.

b) Note that the underlying probability measure is not involved in the definition of the classes $\mathcal{V}^+$ and $\mathcal{V}$.

Let F be a bounded jointly-measurable process and $A \in \mathcal{V}^+$. For almost every ω, $A_\bullet(\omega)$ induces a measure $\mu_A(\omega, ds)$ on $(\mathbb{R}_+, \mathcal{B}_+)$ via

$$\mu_A(\omega,]s,t]) := A_t(\omega) - A_s(\omega), \qquad 0 \leq s < t.$$

As F is bounded and measurable (in t for fixed ω), we can define

$$\int_0^t F_s(\omega)\ dA_s(\omega) := \int F_s(\omega)\ \mu_A(\omega, ds).$$

This works analogously for $A \in \mathcal{V}$. In this case $\mu_A(\omega, ds)$ is in general a signed measure. If the integral is finite for every $t \geq 0$, we write

$$F \bullet A := \int F\ dA$$

for the process $(\int_0^t F_s(\omega)\ dA_s(\omega))_t$ and call this a pathwise Lebesgue-Stieltjes integral. In the case where the integrand process F has continuous paths, the integral can be expressed as the a.s. limit of approximating sums:

Theorem 2.5 *Let $A \in \mathcal{V}$ and H be jointly measurable such that a.s. $s \longmapsto H_s(\omega)$ is continuous. Let π_n be a sequence of partitions of $[0,t]$ with* $\lim_{n\to\infty} \mathrm{mesh}(\pi_n) = 0$. [5] *Then for $t_k \leq s_k \leq t_{k+1}$:*

$$\int_0^t H_s\ dA_s = \lim_{n\to\infty} \sum_{t_k, t_{k+1} \in \pi_n} H_{s_k}(A_{t_{k+1}} - A_{t_k}).$$

Next we prove a change of variables formula for the special case, in which the process A is both: of finite variation and continuous. Itô's formula is a generalization of this result.

[5] $\mathrm{mesh}(\pi_n) = \sup_k\ |t_k - t_{k-1}|$

Theorem 2.6 ***(Change of variables)***
Let A be a finite variation process with continuous paths, and $f \in C^1(\mathbb{R}^d)$. Then $(f(A_t))_{t\geq 0}$ is again of finite variation and

$$f(A_t) - f(A_0) = \int_0^t f'(A_s)\ dA_s\ .$$

Proof: Fix $\omega \in \Omega$. The function $s \longmapsto f'(A_s(\omega))$ is continuous on $[0,t]$, hence bounded. Therefore, the integral $\int_0^t f'(A_s)\ dA_s$ exists.
Fix $t \geq 0$ and let π_n be a sequence of partitions of $[0,t]$ with $\lim_{n\to\infty} \operatorname{mesh}(\pi_n) = 0$. Then

$$\begin{aligned} f(A_t) - f(A_0) \quad &= \sum_{t_k, t_{k+1} \in \pi_n} (f(A_{t_{k+1}}) - f(A_{t_k})) \\ &= \sum_{t_k, t_{k+1} \in \pi_n} f'(A_{\xi_k})(A_{t_{k+1}} - A_{t_k}) \end{aligned}$$

where ξ_k is a point in $[t_k, t_{k+1}]$ (mean value theorem). This sum converges a.s. to $\int_0^t f'(A_s)\ dA_s$ because of Theorem 2.5. □

It is not easy to see, why this formula is called 'change of variables'. The following corollary helps us to understand its meaning.

Corollary 2.7 *Let $g \in C(\mathbb{R}^d)$ and A be of finite variation and continuous. Then*

$$\int_{A_0}^{A_t} g(u)\ du = \int_0^t g(A_s)\ dA_s.$$

Proof: Define $f(t) := \int_0^t g(u)\ du$ and use Theorem 2.6. □

Itô's formula is a generalization of Theorem 2.6. In fact the introduction of a stochastic integral is not straightforward. On the other hand almost all of the interesting processes are of unbounded variation. One can show that an approach using a.s. convergent approximating sums does not work in this context (see [50] Section I.8). K. Itô was the first to introduce a stochastic integral with respect to Brownian motion. This concept was then generalized to martingales and local martingales. This development led to the class of semimartingales which will be introduced in the next section. Before that we have to deal with localization.

In the theory of stochastic integration the following procedure is used time and again:

Definition 2.8 If $\mathcal{C}$ is a class of processes, we denote by $\mathcal{C}_{loc}$ the **localized class**, defined in the following way: a process belongs to $\mathcal{C}_{loc}$ if and only if there exists an increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ (depending on X) such that $T_n \uparrow \infty$ a.s. and each stopped process X^{T_n} belongs to $\mathcal{C}$. See (0.21), (0.22) of the introductory chapter.

Example 2.9 (i) If $\mathcal{M}$ denotes the class of uniformly integrable martingales, then $\mathcal{M}_{loc}$ are the locally (uniformly integrable) martingales. Let us remark that one obtains the same localized class by starting from the (not necessarily uniformly integrable) martingales. [Just set $T_j \wedge j$ if $(T_j)_{j\in\mathbb{N}}$ is a localizing sequence]
We write $M \in \mathcal{M}_{0,loc}$ if $M \in \mathcal{M}_{loc}$ and $M_0 = 0$.

(ii) $\mathcal{V} = \mathcal{V}_{loc}$, i.e. being of finite variation is a local property.

Remarks: a) $\mathcal{C} \subset \mathcal{C}_{loc}$.

b) If $\mathcal{C}$ is stable under stopping, i.e.

$$X \in \mathcal{C}, T \text{ stopping time } \Rightarrow X^T \in \mathcal{C},$$

then $\mathcal{C}_{loc}$ is stable under stopping and $(\mathcal{C}_{loc})_{loc} = \mathcal{C}_{loc}$.

c) If $\mathcal{C}$ and $\mathcal{C}'$ are stable under stopping, then

$$(\mathcal{C} \cap \mathcal{C}')_{loc} = \mathcal{C}_{loc} \cap \mathcal{C}'_{loc}.$$

(See [36] Lemma 1.1.35.)

2.2 Semimartingales and Their Characteristics

Definition 2.10 A stochastic process $X = (X_t)_{t\geq 0}$ is called a **semimartingale** if it has a representation:

$$X_t = X_0 + M_t + A_t$$

where $M \in \mathcal{M}_{0,loc}$ and $A \in \mathcal{V}$.
A d-dimensional process X is called a semimartingale, if every component process $X^{(j)},\ j = 1, ..., d$ is a semimartingale. We write $X \in \mathcal{S}$ respectively $X \in \mathcal{S}^d$ for these classes of processes.

Remarks: a) Note that, since martingales are involved, the notion of semimartingales depends on the underlying probability measure $\mathbb{P}$ as well as on the filtration $\mathbb{F}$.

b) The above defined 'vector of semimartingales' is equivalent to the vector valued semimartingale defined in [46].

c) On the first sight this class of processes seems to be somehow unnatural. Surprisingly enough it turns out that in a certain sense this is the largest class with respect to which stochastic integration is possible (Bichteler-Dellacherie theorem, see [50] chapter III).

Another interesting fact is the wide range of stability properties of this class:

Theorem 2.11 *The space of semimartingales is a vector space, an algebra, a lattice, and is stable under C^2, and more general under convex transformations, i.e. for every f which is convex or a $C^2(\mathbb{R}^d)$-function $(f(X_t))_{t\geq 0}$ is again a semimartingale.*

Proof: See [50], Theorem IV.67. □

In our considerations we combine the notions of Markov processes and semimartingales. Therefore, it is useful to have a concept of processes which are semimartingales with respect to a space $\mathbf{Y} = (\Omega, \mathcal{F}_\infty, \mathbb{F}, (Y_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$. X is a semimartingale w.r.t. $\mathbf{Y}$ if it is a semimartingale w.r.t every $\mathbb{P}^x$ of $\mathbf{Y}$. In the same manner we define martingales and processes of finite variation with respect to $\mathbf{Y}$. If we want to emphasize this dependence we write $\mathcal{S}(\mathbf{Y}), \mathcal{M}(\mathbf{Y})$ and $\mathcal{V}(\mathbf{Y})$.

Definition 2.12 A process X is a **Markov semimartingale** with respect to $\mathbf{Y}$ if it is a semimartingale w.r.t. every $\mathbb{P}^x$ and a Markov process. Analogously we define Hunt semimartingales and Feller semimartingales.

Definition 2.13 We call a process X **additive functional**, if $X_0 = 0$ a.s. and for every $s, t \geq 0$

$$X_{s+t} = X_s + X_t \circ \vartheta_s$$

where ϑ_s is the shift operator introduced in Section 1.2. If a semimartingale (resp. martingale,...) has this property we call it **semimartingale additive functional** (resp. martingale additive functional,...) and we write: $\mathcal{S}_{ad}$ (resp. $\mathcal{M}_{ad}$,...).[6]

All classes of processes we are dealing with are stable under stopping. For semimartingales, however, a stronger localization result holds true: $\mathcal{S}_{loc} = \mathcal{S}$, i.e. A local semimartingale is a semimartingale. To control the range of a jump process it is useful to pre-stop the process instead of stopping it. We introduce this notion in the next definition and remark that this concept is compatible with semimartingales as well.

Definition 2.14 The **pre-stopped process** X^{T-} is defined as follows:

$$X_t^{T-}(\omega) := X_t(\omega) \cdot 1_{[[0,T[[}(\omega, t) + X_{T(\omega)-}(\omega) \cdot 1_{[[T,\infty[[}(\omega, t).$$

We say that a process X is **pre-locally** of class $\mathcal{C}$ if there exists a sequence of stopping times $T_n \uparrow \infty$ a.s. such that $X^{T_n-} \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Theorem 2.15 *A process X is a semimartingale if and only if it is pre-locally a semimartingale.*

Proof: See [50] Theorem II.6. □

The analogous statement for local martingales is wrong, see Example B.1 of the Appendix.

Example 2.16 (i) Brownian Motion is a continuous martingale. Therefore, it is a local martingale and hence in $\mathcal{S}$. If one allows Brownian Motion to start in every $x \in \mathbb{R}^d$, it is a Feller semimartingale.

(ii) A deterministic process $X_t(\omega) = x_t$ is a semimartingale if and only if it is of finite variation and càdlàg. See [36] Proposition I.4.28.

[6] This definition follows [22]. In [13] these classes are called 'additive'.

(iii) Lévy processes are Feller semimartingales by Theorem 1.24.

(iv) A càdlàg process with independent (but not stationary) increments does not have to be a semimartingale. It is one if and only if the generalized first characteristic (see [36] Theorem II.5.2 and Corollary II.5.11) is of finite variation.

We are now in a position to focus on the area of our interest. A Lévy process is both Markovian and a semimartingale. One direction of our investigations is this intersection of process-classes. In Section 3.1 we show that (nice) Feller processes are contained in this intersection.

In order to introduce the **characteristics** of a semimartingale we first need several notions, each of which is interesting on its own. Keep in mind that the definitions depend on the probability measure $\mathbb{P}$ respectively $\mathbb{P}^x$ ($x \in \mathbb{R}^d$). The following theorems are essential (See [36], Theorems I.4.18 and I.4.2).

Theorem 2.17 ***(decomposition of local martingales)***
Any local martingale M admits a unique (up to indistinguishability) decomposition

$$M = M_0 + M^c + M^d$$

where $M_0^c = M_0^d = 0$, M^c is a continuous local martingale, and $M^d \in \mathcal{M}_{0,loc}$ is purely discontinuous, i.e. for every continuous local martingale N we have $M^d \cdot N \in \mathcal{M}_{0,loc}$.

Now we direct our attention to the angle bracket:

Theorem 2.18 *To each pair $M, N \in \mathcal{H}^2_{loc}$ one associates a predictable process $\langle M, N \rangle \in \mathcal{V}$, unique up to an evanescent set, such that $MN - \langle M, N \rangle$ is a local martingale.*

The process $\langle M, N \rangle$ is called the **predictable quadratic covariation** or the **angle bracket**. It has the following properties (see [41] page 211):

Proposition 2.19 ***(properties of the angle bracket)***
Let M, N be locally square integrable martingales and H, K locally bounded and predictable:

a) $(M, N) \longmapsto \langle M, N \rangle$ is bilinear and symmetric.

b) The following polarization identity holds:

$$\langle M, N \rangle = \frac{1}{4}\Big(\langle M+N, M+N \rangle - \langle M-N, M-N \rangle \Big).$$

c) $\langle M, M \rangle \in \mathcal{V}^+$.

d) $\langle M, N \rangle = 0$ if M or N is of finite variation and one of them is continuous.

e) For the stochastic integral which will be introduced in the next section we have

$$\left\langle \int_0^\bullet H_s \, dX_s, \int_0^\bullet K_s \, dY_s \right\rangle_t = \int_0^t H_s K_s \, d\langle X, Y \rangle_s .$$

f) $\langle M^c, M^c \rangle = \langle M, M \rangle^c$.

We introduce the notion of random measures and their compensators.

Definition 2.20 (i) A **random measure** on $\mathbb{R}_+ \times (\mathbb{R}^d \backslash \{0\})$ is a family $\mu = (\mu(\omega; dt, dx))_{\omega \in \Omega}$ of measures on $(\mathbb{R}_+ \times (\mathbb{R}^d \backslash \{0\}), \mathcal{B}_+ \otimes \mathcal{B}(\mathbb{R}^d \backslash \{0\}))$.

(ii) For every $\widetilde{\mathcal{O}}$-measurable mapping $H : \Omega \times \mathbb{R}_+ \times (\mathbb{R}^d \backslash \{0\}) \longrightarrow \mathbb{R}$ and random measure μ we define for $(\omega, t) \in \Omega \times \mathbb{R}_+$:

$$\mu(H)_t(\omega) := \begin{cases} \displaystyle\int_{[0,t]\times(\mathbb{R}^d\backslash\{0\})} H(\omega, s, y)\ \mu(\omega; ds, dy) & \text{if } \displaystyle\int_{[0,t]\times(\mathbb{R}^d\backslash\{0\})} |H(\omega, s, y)|\ \mu(\omega; ds, dy) < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

The stochastic process $\mu(H) := (\mu(H)_t)_{t \geq 0}$ is called the **stochastic integral** of H with respect to μ.

(iii) A random measure is called **predictable**, if it maps $\widetilde{\mathcal{P}}$-measurable functions into $\mathcal{P}$-measurable processes.

(iv) The **compensator** ν of the random measure μ is the $\mathbb{P}$-a.s. unique predictable random measure which fulfills the following property: for every nonnegative $\tilde{\mathcal{P}}$-measurable function H on $\tilde{\Omega}$ we have $\mathbb{E}(\mu(H)_\infty) = \mathbb{E}(\nu(H)_\infty)$.

(v) A function $H : \Omega \times \mathbb{R}_+ \times (\mathbb{R}^d \backslash \{0\}) \longrightarrow \mathbb{R}$ is in the class F_p^j $(j = 1, 2)$ of Ikeda and Watanabe, if it is $\widetilde{\mathcal{P}}$-measurable and for every $t \geq 0$

$$\mathbb{E}\left(\int_0^t \int_{y \neq 0} |H(\cdot, s, y)|^j\ \nu(\cdot; ds, dy)\right) < \infty.$$

We need the following relationship between μ and ν:

Theorem 2.21 *a) If $H \in F_p^1$ we have for every $t \geq 0$*

$$\mathbb{E} \int_0^t \int_{y \neq 0} H(\cdot, s, y)\ \mu(\cdot; ds, dy) = \mathbb{E} \int_0^t \int_{y \neq 0} H(\cdot, ds, dy)\ \nu(\cdot; ds, dy). \tag{2.1}$$

b) If $H \in F_p^2$ then

$$\int_0^t \int_{y \neq 0} H(\cdot, ds, dy)\ \Big(\mu(\cdot; ds, dy) - \nu(\cdot; ds, dy)\Big)$$

is a martingale.

Remark: The last integral cannot be defined directly as the difference of the integrals w.r.t. μ and ν, since both might not even exist. One defines such an integral - with respect to a compensated measure - by using an approximating sequence in $F_p^1 \cap F_p^2$ (see [26]).

Proof: See [26] Section II.3. □

For our purpose it is enough to deal with the random measure, which is given by the jumps: if X is an $\mathbb{R}^d$-valued adapted càdlàg process then

$$\mu^X(\omega; ds, dy) := \sum_{r \geq 0} 1_{\{\Delta X_r(\omega) \neq 0\}} \, \delta_{(r, \Delta X_r(\omega))}(ds, dy) \tag{2.2}$$

where δ is the Dirac measure, defines an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$.

It is well known that for a Lévy process Z the compensator ν of μ^X is deterministic and of the following form:

$$\nu(\cdot; ds, dy) = dsN(dy),$$

where N is the Lévy measure ([26] Example II.4.1). For more information about random measures and their compensators see [36] Section II.1.

We now turn to the so called characteristics of a semimartingale which are in a certain sense a generalization of the Lévy triplet. In the sequel X is an arbitrary semimartingale and $\chi = \chi_R$ is a fixed cut-off function (see Definition 1.4).

Remark: For every cut-off function χ the mapping $y \longmapsto \chi(y) \cdot y$ is a 'truncation function' in the sense of [36]. Both notions are fully compatible.

Next we set:

$$\begin{aligned} \check{X}(\chi) &:= \sum_{0 \leq s \leq \cdot} \Big(\Delta X_s - \chi(\Delta X_s) \Delta X_s \Big) && \in \mathcal{V}^d \\ X(\chi) &:= X - \check{X}(\chi) && \in \mathcal{S}^d. \end{aligned}$$

Then $\Delta X(\chi) = \chi(\Delta X)\Delta X$ which is bounded, hence by [36] Lemma I.4.24 $X(\chi)$ is a special semimartingale. Therefore, it has a (unique) canonical decomposition:

$$X(\chi) = X_0 + M(\chi) + B(\chi), \qquad M(\chi) \in \mathcal{M}^d_{0,loc}, \; B(\chi) \in \mathcal{V}^d \tag{2.3}$$

where $B(\chi)$ is predictable.

Definition 2.22 Let X be a d-dimensional semimartingale and χ a fixed cut-off function. We call **characteristics of X** (with respect to χ) the triplet (B, C, ν) consisting of:

(i) $B = (B^{(j)})_{1 \leq j \leq d}$ is the predictable process $B(\chi)$ appearing in (2.3) above.

(ii) $C = (C^{jk})_{1 \leq j,k \leq d}$ is a continuous process in $\mathcal{V}^{d \times d}$, namely

$$C^{jk} = \left\langle X^{(j),c}, X^{(k),c} \right\rangle$$

where $X^{(j),c}$ is the continuous martingale part of the j-th component (see Theorem 2.17).

(iii) ν is the compensator of the random measure μ^X (see Definition 2.20 (iv) and formula (2.2)).

In the context of Markov processes the characteristics are determined w.r.t. every $\mathbb{P}^x$ ($x \in \mathbb{R}^d$). It is by no means clear that there exist such characteristics which are independent of the starting point x. See [13] Theorem (3.12) for an affirmative answer to this question of existence. Compare in this context our Theorem 3.10.

Example 2.23 Using the Lévy-Itô decomposition (see Theorem 1.24) one obtains for one-dimensional Lévy processes:

$$\begin{aligned} B_t &= \ell t \\ C_t &= \int_0^t \sigma^2 \, d\langle W, W\rangle_s = \sigma^2 t = Qt \\ \nu(\cdot; ds, dy) &= N(dy)\, ds \end{aligned}$$

A particular subclass of semimartingales which is defined by the structure of the characteristics will be interesting for our further investigations (see Section 3.2). It follows directly from the definition that Lévy processes are contained in this class:

Definition 2.24 A semimartingale $X = (X_t)_{t\geq 0}$ is called a **homogeneous diffusion with jumps** if its characteristics are of the form:

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\, ds \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega))\, ds \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy)\, ds \end{aligned} \tag{2.4}$$

where $\ell^{(j)}, q^{jk} : \mathbb{R}^d \longrightarrow \mathbb{R}$ are measurable functions, $Q(x) = (q^{jk}(x))_{1\leq j,k\leq d}$ is a positive semidefinite matrix for every $x \in \mathbb{R}^d$, and $N(x, \cdot)$ is a Borel transition kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d\backslash\{0\})$.

Definition 2.25 Let X be a Hunt semimartingale. We call X an **Itô process** if it is a homogeneous diffusion with jumps (see Definition 2.24 above) with respect to every $\mathbb{P}^x$ ($x \in \mathbb{R}^d$) where ℓ, Q and N are independent of the starting point x. Furthermore, we say that $(\ell, Q, N(\cdot, dy))$ are the **differential characteristics** of the process.

We end this section by giving an interesting characterization of the characteristics:

Theorem 2.26 *Let X be an adapted càdlàg process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and let*

$$B = \tilde{b} \bullet F, \;\; C = \tilde{c} \bullet F, \;\; \nu(\omega; ds, dy) = \tilde{K}(\omega, s; dy)\, dF_s$$

where

(i) $F \in \mathcal{A}^+_{loc}$, and admits a predictable version;

(ii) $\tilde{b} = (\tilde{b}^{(j)})_{1\leq j\leq d}$ is an predictable process;

(iii) $\tilde{c} = (\tilde{c}^{jk})_{1\leq j,k\leq d}$ is an predictable process with values in the symmetric nonnegative matrices;

(iv) $\tilde{K}(\omega, t; dy)$ is a transition kernel from $(\Omega\times\mathbb{R}_+, \mathcal{O})$ into $(\mathbb{R}^d\backslash\{0\}, \mathcal{B}(\mathbb{R}^d\backslash\{0\}))$, satisfying

- $\int_{y\neq 0} (\|y\|^2 \wedge 1)\ \tilde{K}(\omega, s; dy) < \infty$
- $\Delta F_s(\omega) > 0 \Rightarrow \tilde{b}_s(\omega) = \int_{y\neq 0} (\chi(y)y)\ \tilde{K}(\omega, s; dy)$
- $\Delta F_s(\omega)\tilde{K}(\omega, s; \mathbb{R}^d\backslash\{0\}) \leq 1.$

Then the following statements are equivalent:

a) X is a semimartingale and (B, C, ν) are the characteristics of X with respect to the cut-off function χ.

b) For each $\xi \in \mathbb{R}^d$ the process $e^{i\xi' X} - (e^{i\xi X_-}) \bullet A(\xi)$ where

$$A(\xi) = i\xi' B_t - \frac{1}{2}\xi' C_t \xi + \int_{[0,t]\times(\mathbb{R}^d\backslash\{0\})} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y)\right)\ \nu(\cdot; ds, dy)$$

is a complex valued local martingale, i.e. the real and the imaginary part are both local martingales.

c) For every function $u \in C^2(\mathbb{R}^d)$ which is bounded, the process

$$\begin{aligned}\widetilde{M}_t^{[u]} \quad &:= u(X_t) - u(X_0) - \int_0^t \sum_{j=1}^d \partial_j u(X_{s-})\ dB_s^{(j)} \\ &-\frac{1}{2}\int_0^t \sum_{j,k=1}^d \partial_j\partial_k u(X_{s-})\ C_s^{jk} \\ &- \int_0^t \int_{\mathbb{R}^d} \Big(u(X_{s-} + y) - u(X_{s-}) - \chi(y) y' \nabla u(X_{s-})\Big)\ \nu(\cdot; ds, dy)\end{aligned}$$

is a local martingale.

d) For every function $u \in C_b^2(\mathbb{R}^d)$, the process $\widetilde{M}_t^{[u]}$ is a local martingale.

Proof: The properties a)-c) are equivalent by Theorem II.2.42 of [36].

c)⇒d): This implication is trivial, since we are testing against a smaller class of functions in d).

d)⇒b): It is enough to apply c) to the function $u(x) = e^{i\xi' x}$ and to observe that $u(x), \partial_j u(x) = i\xi^{(j)} \cdot e^{i\xi' x}$ and $\partial_j \partial_k = -\xi^{(j)}\xi^{(k)} \cdot e^{i\xi' x}$ are bounded for every $\xi \in \mathbb{R}^d$ and $j, k \in \{1, ..., d\}$. The result follows, since

$$u(X_- + y) - u(X_-) - \sum_{j=1}^d \partial_j u(X_-) y^{(j)} \chi(y) = e^{i\xi' X_-}\left(e^{i\xi' y} - 1 - i\xi' y \chi(y)\right).$$

□

2.3 Stochastic Integration

The notion of a (stochastic) integral should have at least two properties: it should be linear and continuous. Therefore, we introduce the following spaces of stochastic processes on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

- $\mathbb{D}$ = càdlàg, adapted.
- $\mathbb{L}$ = càglàd, adapted.
- $\mathbb{S}$ = simple predictable process, i.e. the process H has a representation: $H_t(\omega) = H_0(\omega)1_{\{0\}} + \sum_{j=1}^{n} \widetilde{H_j}(\omega)1_{]]T_j,T_{j+1}]]}(\omega, t)$ where $\widetilde{H_j}$ is $\mathcal{F}_{T_j}$-measurable and a.s. finite. And $(T_j)_j$ is a finite sequence of finite stopping times such that $T_1 = 0$.

And on these spaces we introduce the **ucp-topology** (= uniformly on compacts in probability):

Definition 2.27 A sequence of processes $(H^n)_{n\in\mathbb{N}}$ **converges ucp** to H if for all $t > 0$

$$\sup_{0\le s\le t} |H_s^n - H_s| \xrightarrow{\mathbb{P}} 0.$$

We write $\mathbb{S}_{ucp}, \mathbb{L}_{ucp}$ and $\mathbb{D}_{ucp}$ for the spaces equipped with this topology.

Remarks: a) The following metric makes $\mathbb{D}_{ucp}$ a complete metric space

$$d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\mathbb{E}(\min\{1, (X-Y)_n^*\}), \qquad X, Y \in \mathbb{D}.$$

b) $\mathbb{S}$ is a dense subset of $\mathbb{L}_{ucp}$.

We first introduce an elementary stochastic integral on $\mathbb{S}$. After that we extend the integral to $\mathbb{L}$ using continuity.

Definition 2.28 Let $H \in \mathbb{S}$ and X be a càdlàg process. We call the linear mapping $J_X : \mathbb{S} \longrightarrow \mathbb{D}$ defined by

$$J_X(H)_t = H_0X_0 + \sum_{j=1}^{n} \widetilde{H_j}\ (X_t^{T_{j+1}} - X_t^{T_j}) \qquad \text{for } t \ge 0$$

the **stochastic integral** of H w.r.t. X (See the representation of H from above).

Note that $X_t^{T_{j+1}} - X_t^{T_j}$ is zero for $t \le T_j$. We use the following three notations interchangeably

$$J_X(H) = \int H\ dX = H \bullet X \tag{2.5}$$

for the process as a whole and

$$J_X(H)_t = \int_0^t H_s\ dX_s = H \bullet X_t \tag{2.6}$$

for the process at time t.

Theorem 2.29 ***(semimartingales are good integrators)***
If X is a semimartingale, then

$$J_X : \mathbb{S}_{ucp} \longrightarrow \mathbb{D}_{ucp}$$

is continuous.

Compare in this context Chapter III of [50].
Now we extend our integral to $\mathbb{L}$. Consider the following diagram:

$$\begin{array}{llll} J_X: & \mathbb{S} & \xrightarrow{\text{cont}} & \mathbb{D}_{ucp} \text{ (complete)} \\ & \cap_{\text{dense}} & \nearrow & \\ & \mathbb{L}_{ucp} & & \end{array}$$

Let us close a small gap in the considerations of P. Protter: the problem is that we can only extend the mapping, if it is uniformly continuous. We can derive this property from the (ordinary) continuity and the translation invariance of the ucp-metric (in both spaces):
Let d resp. d' denote the ucp-metric on $\mathbb{L}$ resp. $\mathbb{D}$ and let $H, K \in \mathbb{S}$. Since J_X is continuous in 0 we have by the linearity of J_X

$$d(H-K,0) < \delta \Rightarrow \quad d'(J_X(H) - J_X(K), 0) = d'(J_X(H-K,0) < \epsilon,$$

but since d and d' are translation invariant, we have

$$d(H,K) < \delta \Rightarrow \quad d'(J_X(H), J_X(K)) < \epsilon.$$

Let us mention that one can extend the space of possible integrands once more: namely to those processes which are **predictable and locally bounded**. The next few theorems are stated for this general case. In our calculations however, most of the integrands will be in $\mathbb{L}$.

Theorem 2.30 *The map J_X has an extension to the space of locally bounded predictable processes $\mathcal{P}_{b,loc}$, which we still denote as in* (2.6). *If $X \in \mathcal{S}$ and $H, K \in \mathcal{P}_{b,loc}$ then the following properties hold:*

(i) $H \bullet J$ is a càdlàg adapted process, i.e. a process in $\mathbb{D}$.

(ii) J_X is linear up to evanescence, i.e. $(\alpha H + \beta K) \bullet X$ and $\alpha(H \bullet X) + \beta(K \bullet X)$ are indistinguishable for $\alpha, \beta \in \mathbb{R}$.

(iii) If a sequence $(H^n)_{n \in \mathbb{N}}$ of predictable processes converges pointwise to a limit H, i.e.

$$\lim_{n\to\infty} H_t^n(\omega) = H_t(\omega) \textit{ for every } (\omega, t) \in \Omega \times \mathbb{R}_+,$$

and if $|H^n| \leq K$ where K is a locally bounded predictable process, then $H^n \bullet X$ converges to $H \bullet X$ in the ucp−topology.

Furthermore, this extension is unique up to evanescence.

Proof: See either [36] pages 46-51 or [50] pages 56-92 and 155-179.

Remark: In a certain sense this is the best result one can get on stochastic integration. If we restrict the integrators to locally square integrable martingales we can enlarge the space of integrands and get better stability results. We do not make use of this concept.

Proposition 2.31 ***(properties of the stochastic integral)***

Let $X \in \mathcal{S}$ and H, K be locally bounded predictable processes. The following equalities (and other statements) hold up to evanescence:

a) $X \longmapsto H \bullet X$ is linear.

b) $H \bullet X$ is a semimartingale.

c) If $M \in \mathcal{M}_{0,loc}$, then $H \bullet M \in \mathcal{M}_{0,loc}$.

d) If $A \in \mathcal{V}$, then $H \bullet A \in \mathcal{V}$ and the stochastic integral coincides with the pathwise Lebesgue-Stieltjes integral introduced in Section 2.1.

e) For every stopping time T we have $(H \bullet X)^T = H1_{[[0,T]]} \bullet X$.

f) $(H \bullet X)_0 = 0$ and $H \bullet X = H \bullet (X - X_0)$.

g) $\Delta(H \bullet X) = H(\Delta X)$.

h) $K \bullet (H \bullet X) = (KH) \bullet X$.

The property c) above is not true for martingales in general. In the next section we will use the square bracket in order to prove a result for martingales.

In the case where the integrand is in $\mathbb{L}$ it is possible to calculate the stochastic integral in terms of approximating sums.

Definition 2.32 (i) We call a finite sequence of finite stopping times

$$0 = T_0 \leq T_1 \leq ... \leq T_k$$

a **random partition**.
(ii) A sequence of random partitions $\sigma_n = (T_0^n \leq ... \leq T_{k_n}^n)$ is said to **tend to the identity** if

- $\lim_{n\to\infty} T_{k_n}^n = \infty$ a.s.
- $\lim_{n\to\infty} \sup_{j=1}^{k_n} \left|T_j^n - T_{j-1}^n\right| = 0$ a.s.

(iii) For $Y \in \mathbb{D}$ or $\mathbb{L}$ we define the process **sampled at** σ by setting

$$Y_t^\sigma(\omega) := Y_0(\omega)1_{\{0\}}(t) + \sum_{j=1}^{k_n} Y_{T_{j-1}}(\omega)1_{]]T_j,T_{j-1}]]}(\omega, t).$$

Obviously Y^σ is a simple predictable process.

Lemma 2.33 *Let $Y \in \mathbb{D}$ or $\mathbb{L}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of random partitions tending to the identity, then*

$$\int_{0+}^{t} Y_s^{\sigma_n} dX_s = \sum_{j=1}^{k_n} Y_{T_j^n} \left(X_t^{T_j^n} - X_t^{T_{j-1}^n}\right) \xrightarrow{ucp} (Y_- \bullet X)_t.$$

2.4 Square Bracket and Martingale Preservation

Let us first introduce the square bracket process of a semimartingale:

Definition 2.34 Let X be a semimartingale. The **quadratic variation process** of X is defined as

$$[X,X] := X^2 - 2\int X_-\, dX.$$

If Y is another semimartingale, then

$$[X,Y] := XY - \int X_-\, dY - \int Y_-\, dX$$

is called the **quadratic covariation** of X and Y or the **square bracket**.

Remark: $(X,Y) \longmapsto [X,Y]$ is bilinear and symmetric. Therefore, we have the following polarization identity:

$$[X,Y] = \frac{1}{4}\Big([X+Y,X+Y] - [X-Y,X-Y]\Big).$$

Next we list some basic properties of the quadratic (co-)variation (see [50] Section II.6).

Proposition 2.35 ***(properties of the square bracket)***
Let $X, Y \in \mathcal{S}$ and H be locally bounded and predictable, then the following properties hold:

a) $[X,X]$ is càdlàg, adapted and increasing.

b) $[X,Y]$ is càdlàg, adapted and of finite variation.

c) $[X,Y]_0 = X_0Y_0$ and in particular $[X,X]_0 = X_0^2$.

d) $\Delta[X,Y] = \Delta X \Delta Y$ and in particular $\Delta[X,X] = (\Delta X)^2$.

e) If $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of random partitions tending to the identity, then

$$X_0^2 + \sum_j \left(X^{T_{j+1}^n} - X^{T_j^n}\right)^2 \xrightarrow{ucp} [X,X].$$

f) If T is a stopping time, then

$$[X^T, X] = [X^T, X^T] = [X,X]^T.$$

g) $XY = \int X_-\, dY + \int Y_-\, dX + [X,Y]$.

h) $[X, H \bullet Y] = \int H\, d[X,Y]$.

Lemma 2.36 $X, Y \in \mathcal{H}^2_{loc}$.
In this case $[X,Y]$ is the unique adapted càdlàg process A, which is of finite variation satisfying:

(i) $XY - A$ is a local martingale.

(ii) $\Delta A = \Delta X \Delta Y$, $A_0 = X_0 Y_0$.

Proof: We follow in this proof mainly [50] (Corollary 2 to Theorem II.27): By Proposition 2.35 g) we can write

$$XY = \int X_- \, dY + \int Y_- dX + [X,Y].$$

Proposition 2.31 c) yields, that the integral terms are local martingales. Thus $XY - [X,Y]$ is a local martingale, too. (ii) follows from 2.35 d). It remains to show uniqueness. Suppose that A and A' both satisfy (i) and (ii). Then

$$A - A' = \underbrace{(XY - A)}_{\in \mathcal{M}_{loc}} - \underbrace{(XY - A')}_{\in \mathcal{M}_{loc}} \in \mathcal{M}_{loc}$$

and

$$\Delta(A - A') = \Delta A - \Delta A' = \Delta X \Delta Y - \Delta X \Delta Y = 0.$$

Thus $A - A'$ is a continuous local martingale in $\mathcal{V}$ with the property $A_0 - A'_0 = 0$. By [36] Theorem I.3.16 the process $A - A'$ is constantly zero. $\square$

Since the process $[X,X]$ is non-decreasing with right-continuous paths, we can decompose the process path-by-path into its continuous part and its pure jump part.

Definition 2.37 The process $[X,X]^c$ denotes the path-by-path **continuous part** of $[X,X]$. We can write:

$$[X,X]_t = [X,X]^c_t + \sum_{0 \leq s \leq t} (\Delta X_s)^2.$$

The same decomposition is used on $[X,Y]$ to obtain $[X,Y]^c$. A semimartingale X with the property $[X,X]^c \equiv 0$ will be called **quadratic pure jump**.

Remarks: a) If $A \in \mathcal{V}$ then A is quadratic pure jump.

b) If X is quadratic pure jump and $Y \in \mathcal{S}$, then $[X,Y] = \sum_{0 \leq s \leq t} \Delta X_s \Delta Y_s$. Therefore, we have $[X,Y]^c \equiv 0$ in this case (see [50] Theorem II.28).

Example 2.38 (One-dimensional)
Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a deterministic function which is of finite variation on compacts, $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ a continuous function, $A \in \mathcal{V}$, W be a standard Brownian motion, and Z a Lévy process with characteristic triplet (ℓ, Q, N), then

(i) $[f, g]_t = 0$

(ii) $[A, A]_t = \sum_{0 \leq s \leq t} (\Delta A_s)^2$

(iii) $[W, W]_t = t$

(iv) $[W, A]_t = 0$

(v) $[Z, Z]_t = Qt + \sum_{0 \le s \le t} (\Delta Z_t)^2$.

In particular we have $[Z, Z]_t^c = Qt$.

There is a close relationship between the square bracket and the angle bracket (see Section 2.2) which reads as follows:

Theorem 2.39 *a) If $X, Y \in \mathcal{S}$ and if X^c resp. Y^c denote their continuous martingale parts, then*

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{0 \le s \le t} \Delta X_s \Delta Y_s.$$

We have in particular

$$[X^c, Y^c] = \langle X^c, Y^c \rangle = [X, Y]^c.$$

b) If $X, Y \in \mathcal{H}^2_{loc}$ then $[X, Y] \in \mathcal{A}_{loc}$ and its predictable compensator is $\langle X, Y \rangle$, i.e.

$$[X, Y] - \langle X, Y \rangle \ \text{ is a local martingale.}$$

Proof: See [36] Theorem I.4.52 and Proposition I.4.50 b). □

Example 2.40 To emphasize the difference between the square bracket and the angle bracket, it is useful to investigate Poisson processes. Let N be a **standard Poisson process** with rate $\lambda (> 0)$, i.e. a one-dimensional Lévy process with characteristic exponent $\psi(\xi) = \lambda(1 - \exp(i\xi))$ (for more information about this class of processes see e.g. [11] Chapter 4). It is well known that $\tilde{N}_t := N_t - \lambda t$ is a martingale. For this process we obtain

$$\left[\tilde{N}, \tilde{N}\right]_t = N_t \qquad \text{but} \qquad \left\langle \tilde{N}, \tilde{N} \right\rangle_t = \lambda t$$

and

$$\left\langle \tilde{N}^c, \tilde{N}^c \right\rangle_t = 0 \ne \lambda t = \left\langle \tilde{N}, \tilde{N} \right\rangle_t^c$$

where the angle bracket has been decomposed in an analogous manner as the square bracket (see Definition 2.37).

Coming back to the martingale preservation we cite the following theorem which is taken from [50] (Corollary II.3). Its corollary will be used in Section 5.1.

Theorem 2.41 *let $M \in \mathcal{M}_{0,loc}$. Then M is a martingale such that $\mathbb{E}M_t^2 < \infty$ for all $t \ge 0$ if and only if $\mathbb{E}\,[M, M]_t < \infty$ for all $t \ge 0$.*

Corollary 2.42 ***(martingale preservation)***

Let M be a martingale such that $\mathbb{E}M_t^2 < \infty$ for all $t \ge 0$ and H be a bounded process of class $\mathbb{L}$. Then the process $H \bullet M$ is a martingale and $\mathbb{E}(H \bullet M)_t < \infty$ for all $t \ge 0$.

Proof: Let H be bounded by $c \in \mathbb{R}_+$.

$$\mathbb{E}[H \bullet M, H \bullet M]_t = \mathbb{E} \int_0^t H_s^2 d[M, M]_s \le \mathbb{E}c^2 \int_0^t d[M, M]_s \le c^2 \mathbb{E}[M, M]_t < \infty$$

and the assertion follows by Theorem 2.41. □

2.5 Itô's Formula

Itô's Formula is one of the main tools which will be used in the following. Therefore, we state it in two different versions.

Theorem 2.43 ***(Itô)***
Let X be a semimartingale and $f \in C^2(\mathbb{R}, \mathbb{R})$. Then $f(X)$ is again a semimartingale and the following formula holds:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0+}^{t} f'(X_{s-})\, dX_s + \frac{1}{2}\int_{0+}^{t} f''(X_{s-})\, d[X,X]_s^c \\ &\quad + \sum_{0<s\le t} \Big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s \Big) \end{aligned} \tag{2.7}$$

where the $0+$ indicates that we are integrating on $]0,t]$. This formula is also true for complex valued holomorphic functions f.

Remarks: a) Formula (2.7) is equivalent to

$$\begin{aligned} f(X_t) - f(X_0) = &\int_{0+}^{t} f'(X_{s-})\, dX_s + \frac{1}{2}\int_{0+}^{t} f''(X_{s-})\, d[X,X]_s \\ &+ \sum_{0<s\le t} \Big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_{s-})(\Delta X_s)^2 \Big), \end{aligned}$$

since

$$[X,X]_t - [X,X]_t^c = \sum_{0\le s\le t} (\Delta X_s)^2$$

and we are integrating on $]0,t]$.

b) In the literature one also finds the different looking formula (see e.g. [36]):

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0}^{t} f'(X_{s-})\, dX_s + \frac{1}{2}\int_{0}^{t} f''(X_{s-})\, d[X,X]_s^c \\ &\quad + \sum_{0\le s\le t} \Big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s \Big). \end{aligned}$$

However, the jump at zero which is added here in the first term is subtracted in the third term. And since we are integrating with respect to a continuous process of finite variation in the second term it makes no difference whether we are integrating on $]0,t]$ or on $[0,t]$.

Proof: See [50] Theorem II.32 for the case of differentiable f and Theorem II.36 for the analytic case. $\square$

Finally we state the multidimensional version of Itô's formula:

Theorem 2.44 ***(Itô)***
Let X be a d-dimensional semimartingale and let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ have continuous second order partial derivatives. Then $f(X)$ is a semimartingale and the following formula holds:

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{j=1}^{d} \int_{0+}^{t} \partial_j f(X_{s-})\, dX_s^{(j)} + \frac{1}{2} \sum_{j,k=1}^{d} \int_{0+}^{t} \partial_j \partial_k f(X_{s-})\, d[X^{(j)}, X^{(k)}]_s^c \\ &\quad + \sum_{0<s\le t} \left(f(X_s) - f(X_{s-}) - \sum_{j=1}^{d} \partial_j f(X_{s-}) \Delta X_s^{(j)} \right). \end{aligned}$$

This formula is also true for complex valued holomorphic functions f.

Proof: See [50] Theorem II.33 for the real valued case. The analytic case works alike.□

2.6 Stochastic Differential Equations

Most of the results presented here can be found in P. Protter's monograph [50] in Chapter V.

In this section we will deal with stochastic differential equations (SDEs) of the following kind

$$X_t = x + \int_0^t \Phi(X_{s-})\, dY_s \tag{2.8}$$

where $(Y_t)_{t\ge 0}$ is a semimartingale, Φ the so called **coefficient** and we have a single probability measure $\mathbb{P}$ in the background. The equation (2.8) is meant either one-dimensional or in a 'matrix and vector'-sense. In the latter case it stands for the system of equations

$$X_t^{(j)} = x^{(j)} + \sum_{k=1}^{n} \int_0^t \Phi^{jk}(X_{s-})\, dY_s^{(k)}, \qquad 1 \le j \le d$$

where $Y = (Y^{(1)}, .., Y^{(n)})' \in \mathcal{S}^n$ and $\Phi(x) = (\Phi^{jk}(x))_{1\le j\le d, 1\le k\le n}$.

The $x \in \mathbb{R}^d$ in (2.8) is an initial condition. Obviously the solution of the SDE will depend on x. Whether or not this happens in a continuous way is interesting in the context of Feller processes (see below). We formalize this concept in the following way: write $X(\omega, t, x) = X_t^x(\omega)$ for the solution of (2.8) depending on x. The **flow** of the SDE is the mapping

$$\varphi : \mathbb{R}^d \longrightarrow D(\mathbb{R}_+, \mathbb{R}^d) : x \longmapsto X(\omega, \cdot, x)$$

defined for every ω. The function space $D(\mathbb{R}_+, \mathbb{R}^d)$ of càdlàg functions is equipped with the topology of uniform convergence on compacts.

The following general result holds true:

Theorem 2.45 *Let the coefficient Φ be locally Lipschitz continuous, i.e. there exists an increasing sequence of open sets Λ_m such that $\cup_{m\in\mathbb{N}}\Lambda_m = \mathbb{R}^d$ and Φ is Lipschitz on Λ_m for every $m \in \mathbb{N}$. Then there exists a function $\zeta : \mathbb{R}^d \times \Omega \longrightarrow [0, \infty]$ such that $\zeta(x, \cdot)$ is a stopping time for each $x \in \mathbb{R}^d$, and there exists a unique (strong) solution of the SDE (2.8) up to $\zeta(x, \cdot)$ and the flow φ of X is continuous on $[[0, \zeta(x, \cdot)[[$.*

Proof: See [50] Theorem V.38. □

Remarks: a) For each fixed x the stopping time $\zeta(x,\cdot)$ is called an **explosion time**, since

$$\limsup_{t\uparrow\zeta(x,\cdot)} \|X_t^x\| = \infty \text{ a.s.}$$

We are interested in conservative solutions, i.e. $\zeta = \infty$ a.s. A sufficient criterion to obtain this is that the coefficient is (globally) Lipschitz continuous.

b) As locally uniform convergence implies pointwise convergence, we have continuity ($\mathbb{R}^d \longrightarrow \mathbb{R}^d$) for a fixed $t \geq 0$ as well. For our purposes this continuity is always sufficient.

Another interesting point is the relationship between Lévy processes as driving terms and Markov processes as solutions. Roughly speaking: if the driving term in (2.8) is a Lévy process then the solution is strongly Markovian and time-homogeneous and the converse is also true.

A small technical difficulty arises if one takes the starting point into account; at least if all the processes X^x should be defined on the same probability space. The original space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the driving Lévy process is defined might be too small as a source of randomness for the solution. We overcome this problem by setting:

$$\begin{aligned} \widetilde{\Omega} &:= \mathbb{R}^d \times \Omega \\ \widetilde{\mathcal{F}_t^0} &:= \mathcal{B}^d \otimes \mathcal{F}_t \qquad\qquad\qquad \widetilde{\mathcal{F}_t} := \cap_{u>t} \widetilde{\mathcal{F}_u^0} \\ \mathbb{P}^x &:= \delta_x \times \mathbb{P} \qquad \text{for every } x \in \mathbb{R}^d. \end{aligned} \tag{2.9}$$

A random variable Z defined on Ω is considered to be extended automatically to $\widetilde{\Omega}$ by $Z(\widetilde{\omega}) = Z(\omega)$, for $\widetilde{\omega} = (y, \omega)$.

Let us have a closer look at SDEs which are driven by Lévy processes. We will come back to this class of processes in Section 5.1, where we calculate the symbol associated to the solution of such SDEs.

Theorem 2.46 *Let Z be an n-dimensional Lévy processes such that $Z_0 = 0$ and Φ be globally Lipschitz. In this case the solution of*

$$X_t = x + \int_0^t \Phi(X_{s-})\, dZ_s \tag{2.10}$$

is a simple Markov process w.r.t. every $\mathbb{P}^x$, $x \in \mathbb{R}^d$.

Proof: The simple Markov property (0.37) follows from [50] Theorem V.32. Note that P. Protter formulates the theorem only for the special case where the components of the process are independent (cf. Proposition 1.26). However, the independence is not used in the proof. □

Although this is already an interesting result, it is not sufficient for our purposes: it could happen that the process is a simple strong Markov process with respect to every $\mathbb{P}^x$, but with different transition kernels for different x, i.e. writing

$$P^y_{s,s+t}(x, B) = \mathbb{P}^y(X_{s+t} \in B | X_s = x)$$

one could find $r, s, t \geq 0$, $w, x, y \in \mathbb{R}^d$ and $B \in \mathcal{B}^d$ such that

$$P^y_{s,s+t}(x, B) \neq P^w_{r,r+t}(x, B).$$

As an example take the Markov processes given by the above theorem (w.r.t. a non-trivial driving process) and replace X^0 by the process which is identically zero. This is still a simple Markov process with respect to every $\mathbb{P}^x$, but with different transition probabilities if it starts in zero. Compare in this context Example B.2 of the appendix. There we investigate the solution of an SDE driven by a two-dimensional process which is not a Lévy process, but has one-dimensional Lévy components. In this case the solution does not have to be time-homogeneous any more. It is the content of the next theorem, that problems like this do not appear if the driving process is a d-dimensional Lévy process

Theorem 2.47 *In the setting of the Theorem 2.45 the transition functions are the same for every $\mathbb{P}^x$ $(x \in \mathbb{R}^d)$. In particular we can set*

$$P_t(x, B) := P_{0,t}(x, B) := P^x_{0,t}(x, B)$$

and the process X associated to this transition function is a universal Markov process.

Proof: We have to show that for arbitrary $x, y \in \mathbb{R}^d$, $s, t \geq 0$ and $B \in \mathcal{B}^d$ the following equation holds:

$$P^x_{0,t}(x, B) = P^y_{s,s+t}(x, B) \tag{2.11}$$

where the superscript indicates with respect to which probability measure the transition kernel is defined. We fix $x, y \in \mathbb{R}^d$. Recall that X^x is the solution of the SDE

$$X_t = x + \int_0^t \Phi(X_{r-})\, dZ_r$$

under the probability measure $\mathbb{P}^x$. Now fix an $s \geq 0$ which satisfies $X^y_s = x$ (Note, that there might be an x which is not in the range of X^y_s. In this case we can set $P^y_{s,s+t}(x, B)$ equal to an arbitrary value). Under $\mathbb{P}^y(\cdot | X_s = x)$ we have

$$X_{s+t} = y + \int_0^{s+t} \Phi(X_{r-})\, dZ_r = \underbrace{y + \int_0^s \Phi(X_{r-})\, dZ_r}_{=X^y_s = x} + \int_s^{s+t} \Phi(X_{r-})\, dZ_r.$$

By definition the process $(X^y_{s+t})_{t\geq 0}$ solves this equation which is in the Lévy driven case equivalent to

$$Y_t = x + \int_0^t \Phi(Y_{r-})\, dZ_r \qquad \text{under } \mathbb{P}^x.$$

And the unique solution of this SDE is X^x_t. Therefore, we obtain

$$\mathbb{P}^y(X_{s+t} \in B | X_s = x) = \mathbb{P}^x(X_t \in B).$$

Furthermore, Theorem V.31 of [50] ensures the existence of a version of X satisfying (MP2). □

It is interesting that the converse is also true, at least, if the coefficient Φ is never zero, i.e. $\{y \in \mathbb{R}^d : \Phi(y) = 0\} = \emptyset$.

Theorem 2.48 *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space with* $Z \in \mathcal{S}$*. Let* $\Phi \in B(\mathbb{R})$ *such that* Φ *is never zero and is such that for every* $x \in \mathbb{R}$ *the equation* (2.8) *has a unique (strong) solution* X^x*. If each of the processes* X^x *is a time-homogeneous Markov process with the same transition semigroup, i.e. equation* (2.11) *holds, then* Z *is a Lévy process.*

Proof: We follow mainly the proof of [38] Theorem 1, but compare in this context also [50] Theorem V.73. Let X denote the canonical process on $D(\mathbb{R}_+, \mathbb{R})$ and let $\mathbb{P}^x$ denote the law of X^x on the canonical space. Under our hypotheses the process

$$(D(\mathbb{R}_+, \mathbb{R}), \mathcal{F}^0_\infty, \mathbb{F}^0, (\vartheta_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}}$$

is a universal Markov process.
Since Φ is never zero, we have

$$Z_t = Z_0 + \int_0^t \Phi(X^x_{s-})^{-1}\Phi(X^x_{s-})\, dZ_s = Z_0 + \int_0^t \Phi(X^x_{s-})^{-1}\, dX^x_s$$

with respect to $\mathbb{P}$ for every x, hence

$$Z_t - Z_0 = \int_0^t \Phi(X^x_{s-})^{-1}\, dX^x_s.$$

On $D(\mathbb{R}_+, \mathbb{R})$ we define

$$\tilde{Z}_t := \int_0^t \Phi(X_{s-})^{-1}\, dX_s$$

with respect to $\mathbb{P}^x$ $(x \in \mathbb{R})$. One obtains that the finite dimensional distributions of $(\tilde{Z}_t)_{t\geq 0}$ under $\mathbb{P}^x$ are the same as the finite dimensional distributions of $(Z_t - Z_0)_{t\geq 0}$ under $\mathbb{P}$, i.e. for $x \in \mathbb{R}$, $n \in \mathbb{N}$, $t_1 < ... < t_n$ and $B_1, ..., B_d \in \mathcal{B}^1$ we have

$$\mathbb{P}^x(\tilde{Z}_{t_1} \in B_1, ..., \tilde{Z}_{t_n} \in B_n) = \mathbb{P}(Z_{t_1} - Z_0 \in B_1, ..., Z_{t_n} - Z_0 \in B_n). \tag{2.12}$$

On the other hand $\tilde{Z}$ is an additive functional. For every positive Borel function g the Markov property and (2.12) imply

$$\begin{aligned} \mathbb{E}^x(g(\tilde{Z}_{t+s} - \tilde{Z}_t)|\mathcal{F}^0_t) &= \mathbb{E}^x(g((\tilde{Z}_s - \tilde{Z}_0) \circ \vartheta_t)|\mathcal{F}^0_t) \\ &= \mathbb{E}^x(g(\tilde{Z}_s \circ \vartheta_t)|\mathcal{F}^0_t) \\ &= \mathbb{E}^{X_t}(g(\tilde{Z}_s)) \\ &= \mathbb{E}(g(Z_s - Z_0)). \end{aligned}$$

Therefore, $\tilde{Z}_{t+s} - \tilde{Z}_t$ is independent of $\mathcal{F}^0_t$ hence of every increment of $\tilde{Z}$ before time t. And the law of $\tilde{Z}_{t+s} - \tilde{Z}_t$ under $\mathbb{P}^x$ is the same as the law of $Z_s - Z_0$ under $\mathbb{P}$ which in turn is the same as the law $\tilde{Z}_s$ under $\mathbb{P}^x$ by (2.12). Using again (2.12) we obtain that the properties (L1) and (L2) are fulfilled by Z and since the process is càdlàg it is a Lévy process. □

Theorem 2.49 *Let Φ be bounded and Lipschitz continuous. In this case the solution X_t^x of the SDE* (2.10)

$$X_t = x + \int_0^t \Phi(X_{s-})\, dZ_s$$

is a Feller process.

Proof: The Markov property follows from Theorem 2.46 and Theorem 2.47. It remains to show the properties (g) and (h) of Definition 1.19.
Ad (h): Let $u \in C_\infty(\mathbb{R}^d),\ t \geq 0$ and consider

$$|\mathbb{E}^x u(X_t) - \mathbb{E}^y u(X_t)| = |\mathbb{E}(u(X_t^x) - u(X_t^y))| =: (\star).$$

Using a continuity-of-the-flow result (see Theorem 2.45) we obtain that for almost every $\omega \in \Omega:\ \lim_{y\to x} X_t^y(\omega) = X_t^x(\omega)$ uniformly on compacts. For these ω's we have: $u(X_t^y(\omega)) \longrightarrow u(X_t^x(\omega))$ and as $|u(X_t^x) - u(X_t^y)| \leq 2\,\|u\|_\infty$ which is finite, because u is bounded, we obtain that $(\star)$ tends to 0 as $y \to x$ by Lebesgue's theorem.
Fix $u \in C_\infty(\mathbb{R}^d)$ and $t \geq 0$. We now have to show

$$T_t u(x) = \mathbb{E}^x u(X_t) \underset{\|x\|\to\infty}{\longrightarrow} 0$$

i.e. for every $\varepsilon > 0$ there exists a compact set $\tilde{K}$ such that on the complement $\tilde{K}^c$ of this set we have

$$|\mathbb{E}^x u(X_t)| = |\mathbb{E} u(X_t^x)| = \left|\mathbb{E} u \left(x + \int_0^t \Phi(X_{s-}^x)\, dZ_s\right)\right| < \varepsilon.$$

In fact this is the crucial point of the proof. Let $\widetilde{X}$ denote the solution of the modified SDE

$$\widetilde{X}_t = x + \int_0^t \Phi(\widetilde{X}_{s-})\, d\widetilde{Z}_s$$

where $\widetilde{Z}_t = Z_t - \sum_{0<s\leq t}(\Delta Z_s 1_{\{|\Delta Z_s|\geq 1\}})$ is a Lévy process with bounded jumps. First we show that for fixed $\delta, \varepsilon > 0$ there exists a radius $k > \delta > 0$ such that

$$\|x\| \geq k \qquad \Rightarrow \qquad \mathbb{P}^x\left(\left\|\widetilde{X}_t\right\| < \delta\right) < \varepsilon. \tag{2.13}$$

To this end we consider for $\|x\| > \delta$

$$\begin{aligned}
\mathbb{P}^x\left(\left\|\widetilde{X}_t - 0\right\| < \delta\right) &\leq \mathbb{P}^x\left(\left\|\widetilde{X}_t - x\right\| > \|x\| - \delta\right) \\
&= \mathbb{P}^x\left(\left\|(\Phi(\widetilde{X}_-) \bullet \widetilde{Z})_t\right\| > \|x\| - \delta\right) \\
&= \mathbb{P}^x\left(\left\|(\Phi(\widetilde{X}_-) \bullet \widetilde{Z})_t\right\|^2 > (\|x\| - \delta)^2\right) \\
&\leq \frac{\mathbb{E}^x\left(\left\|(\Phi(\widetilde{X}_-) \bullet \widetilde{Z})_t\right\|^2\right)}{(\|x\| - \delta)^2}
\end{aligned}$$

and show that the numerator is uniformly bounded in x. For the sake of readability we do this only in the one-dimensional case. In the multi-dimensional case one has to consider finite sums. We denote by $\widetilde{Z}^t$ the process stopped at the deterministic time t. By our assumption the integral $\Phi(\widetilde{X}_-) \bullet \widetilde{Z}^t$ has bounded jumps and is hence a special semimartingale. Since we have (where M denotes the martingale part of $\widetilde{Z}^t$)

$$\begin{aligned}
\left\| \Phi(\widetilde{X}_-) \bullet \widetilde{Z}^t \right\|_{\mathcal{H}^2} &:= \left\| \left[\Phi(\widetilde{X}_-) \bullet M, \Phi(\widetilde{X}_-) \bullet M \right]_t^{1/2} \right\|_{L^2} + \left\| \int_0^t \left| d(\Phi(\widetilde{X}_{s-}) \bullet \ell s) \right| \right\|_{L^2} \\
&\leq \left\| \int_0^t \left| \Phi(\widetilde{X}_{s-}) \right|^2 Q \, ds \right\|_{L^2} + \left\| \int_0^t \left| \Phi(\widetilde{X}_{s-}) \right| |\ell| \, ds \right\|_{L^2} \\
&\leq \left(\|\Phi\|_\infty^2 tQ \right)^{1/2} + \|\Phi\|_\infty t \, |\ell| \\
&< \infty
\end{aligned}$$

it is even an $\mathcal{H}^2$-semimartingale in the sense of P. Protter (ℓ and Q are as defined in Theorem 1.24). We obtain by Theorem IV.5 of [50]

$$\begin{aligned}
\mathbb{E}^x \left| \left(\Phi(\widetilde{X}_-) \bullet \widetilde{Z} \right)_t \right|^2 &\leq \mathbb{E}^x \left(\sup_{0 \leq s \leq t} \left| (\Phi(\widetilde{X}_-) \bullet \widetilde{Z})_s \right| \right)^2 \\
&= \mathbb{E}^x \left(\sup_{s \geq 0} \left| (\Phi(\widetilde{X}_-) \bullet \widetilde{Z}^t)_s \right| \right)^2 \\
&\leq 8 \left\| \Phi(\widetilde{X}_-) \bullet \widetilde{Z}^t \right\|_{\mathcal{H}^2} \\
&\leq 8 \left(\|\Phi\|_\infty^2 tQ \right)^{1/2} + 8 \|\Phi\|_\infty t \, |\ell|
\end{aligned}$$

which is finite and does not depend on x. We have thus established (2.13).
Now we fix $\varepsilon > 0$. Since $u \in C_\infty(\mathbb{R}^d)$ there exists a $\delta > 0$ such that $\|y\| > \delta$ implies $|u(y)| < \varepsilon/2$. By our above considerations there exists a $k > \delta > 0$ such that

$$\|x\| \geq k \qquad \Rightarrow \qquad \mathbb{P}^x \left(\left\| \widetilde{X}_t \right\| < \delta \right) < \frac{\varepsilon}{2 \|u\|_\infty}.$$

Therefore, we obtain for the semigroup $\widetilde{T}_t$ associated to $\widetilde{X}$ that for $\|x\| > k$:

$$\begin{aligned}
\left| \widetilde{T}_t u(x) \right| &\leq \int_{\mathbb{R}^d} |u(y)| \, \widetilde{p}_t(x, dy) \\
&= \int_{B_\delta(0)} |u(y)| \, \widetilde{p}_t(x, dy) + \int_{B_\delta(0)^c} |u(y)| \, \widetilde{p}_t(x, dy) \\
&\leq \|u\|_\infty \cdot \mathbb{P}^x(\widetilde{X}_t \in B_\delta(0)) + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

The general case then follows by an interlacing argument (cf. [2] Theorem 6.2.9). We have thus established (h).
Ad (g): Knowing that (h) holds, it is enough to prove pointwise convergence in the setting of (g) (see e.g. [52] Proposition III.(2.4)). Therefore, we fix a $u \in C_\infty(\mathbb{R}^d)$ and let $x \in \mathbb{R}^d$. Since the paths of X^x are càdlàg we have

$$X_t^x(\omega) \underset{t \to 0}{\longrightarrow} X_0^x(\omega) \qquad a.s.$$

Using the dominated convergence theorem again we obtain at once

$$\lim_{t\to 0} T_t u(x) = T_0 u(x) = u(x)$$

and the assertion follows. □

Remark: A different way to prove that $\widetilde{T}_t u$ vanishes at infinity, is by using the estimate

$$\mathbb{E}\Big((1 + \left\|\widetilde{X}_t^x - \widetilde{X}_s^x\right\|^2)^p\Big) \le C_{p,t}(1 + \|x\|^2)^p$$

for all $0 \le s \le t < \infty$, $x \in \mathbb{R}^d$, $p \in \mathbb{R}$ and a suitable constant $C_{p,t} > 0$. (cf. [2] Theorem 6.7.2 and [21]).

Next we show that the solution of the SDE (2.10) is nice if Φ is bounded, or if a weaker 'growth condition' is satisfied (see the remark below Theorem 2.50).

Theorem 2.50 *Let Φ be bounded and Lipschitz continuous. In this case the solution X_t^x of the SDE* (2.10)

$$X_t = x + \int_0^t \Phi(X_{s-})\, dZ_s$$

is nice, i.e the test functions are contained in the domain $D(A)$ of the generator A.

Proof: We give the one-dimensional proof. The multi-dimensional version works alike. Let $u \in C_c^\infty(\mathbb{R})$ and consider

$$\frac{\mathbb{E}^x u(X_t) - u(x)}{t} = \frac{1}{t}\mathbb{E}^x(u(X_t) - u(x)) =: (\star).$$

Now we use Itô's formula (2.7) for the function u:

$$\begin{aligned}(\star) = \frac{1}{t}\mathbb{E}^x \Bigg(&\int_{0+}^t u'(X_{s-})\, dX_s + \frac{1}{2}\int_{0+}^t u''(X_{s-})\, d[X,X]_s^c \\ &+ \sum_{0<s\le t} (u(X_s) - u(X_{s-}) - u'(X_{s-})\Delta X_s)\Bigg).\end{aligned}$$

Since $X = x + \Phi(X_-) \bullet Z$ we obtain by Propositions 2.30 g),h) and 2.34 g)

$$\begin{aligned}(\star) = &\frac{1}{t}\mathbb{E}^x \Bigg(\int_{0+}^t u'(X_{s-})\Phi(X_{s-})\, dZ_s + \frac{1}{2}\int_{0+}^t u''(X_{s-})\Phi(X_{s-})^2\, d[Z,Z]_s^c \\ &+ \frac{1}{t}\mathbb{E}^x \int_{y\ne 0}\int_0^t \Big(u(X_{s-} + \Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y\Big)\ \mu^Z(\cdot; ds, dy)\end{aligned}$$

where we wrote the last term as an integral with respect to the jump measure μ^Z. Next we use the Lévy-Itô decomposition (Theorem 1.24) in the first term. The expected value of the integral with respect to the martingale part of Z is constantly zero, since the integral

$$u'(X_{s-})\Phi(X_{s-}) \bullet \left(\sigma W_t + \int_{[0,t]\times\{|y|<1\}} y\ \Big(\mu^Z(ds,dy) - ds N(dy)\Big)\right)$$

is a martingale by Corollary 2.42. The square bracket of a Lévy process was calculated in Example 2.38 (v).

$$(\star) = \frac{1}{t}\mathbb{E}^x \int_{0+}^t u'(X_{s-})\Phi(X_{s-})\, d\left(\ell t + \sum_{0<r\le s} \Delta Z_r 1_{\{|\Delta Z_r|\ge 1\}}\right)$$
$$+\frac{1}{2}\frac{1}{t}\mathbb{E}^x \int_{0+}^t u''(X_{s-})\Phi(X_{s-})\, d(\sigma^2 s)$$
$$+\frac{1}{t}\mathbb{E}^x \int_{y\neq 0}\int_0^t \Big(u(X_{s-}+\Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y\Big)\ \mu^Z(\cdot; ds, dy)$$

We write the jump part of the first term as an integral with respect to μ^Z and add it to the third term. Next we show that the integrand

$$H(\cdot, s, y) := u(X_{s-}+\Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y 1_{\{|y|<1\}}$$

is in the class F_p^1 of Ikeda and Watanabe (see Definition 2.20 (v)), i.e. it is $\widetilde{\mathcal{P}}$-measurable and

$$\mathbb{E}\left(\int_0^t \int_{y\neq 0} |H(\cdot, s, y)|\, \nu(\cdot, ds, dy)\right) < \infty$$

where ν denotes the compensator of μ^X. The measurability condition is fulfilled because of the left-continuity of the paths and the integrability follows from

$$\begin{aligned}
&\left|u(X_{s-}+\Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y1_{\{|y|<1\}}\right| \\
&\quad\le \left|1_{\{|y|<1\}}\cdot u(X_{s-}+\Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y\right| + 1_{\{|y|\ge 1\}} 2\left\|u\right\|_\infty \\
&\quad\le 1_{\{|y|<1\}} y^2 \Phi(X_{s-})^2 \left\|u''\right\|_\infty + 1_{\{|y|\ge 1\}} 2\left\|u\right\|_\infty \\
&\quad\le (2\vee \left\|\Phi\right\|_\infty^2)(y^2\wedge 1)(\left\|u\right\|_\infty + \left\|u''\right\|_\infty)
\end{aligned}$$

where we used a Taylor expansion in the first term. Therefore, $H\in F_p^1$ and we can integrate with respect to the compensator of the random measure instead of the measure itself 'under the expectation' by Theorem 2.21:

$$(\star) = \frac{1}{t}\mathbb{E}^x \int_{0+}^t u'(X_{s-})\Phi(X_{s-})\ell\, ds + \frac{1}{2}\frac{1}{t}\mathbb{E}^x \int_{0+}^t u''(X_{s-})\Phi(X_{s-})\sigma^2\, ds$$
$$+\frac{1}{t}\mathbb{E}^x \int_{y\neq 0}\int_0^t \Big(u(X_{s-}+\Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y1_{\{|y|<1\}}\Big)\ dsN(dy).$$

Since we are integrating with respect to Lebesgue measure and since the paths of a càglàd process have only a countable number of jumps we can write:

$$(\star) = \frac{1}{t}\mathbb{E}^x \int_0^t u'(X_s)\Phi(X_s)\ell\, ds + \frac{1}{2}\frac{1}{t}\mathbb{E}^x \int_0^t u''(X_s)\Phi(X_s)\sigma^2\, ds$$
$$+\frac{1}{t}\mathbb{E}^x \int_0^t\int_{y\neq 0} \Big(u(X_s+\Phi(X_s)y) - u(X_s) - u'(X_s)\Phi(X_s)y1_{\{|y|<1\}}\Big)\ N(dy)ds.$$

The possibility to change the order of integration is again given by the estimate of $|H|$ above. Finally we obtain by Lemma 2.51 a) below that

$$\frac{\mathbb{E}^x u(X_t) - u(x)}{t} \xrightarrow[t\downarrow 0]{} \ell u'(x)\Phi(x) + \frac{1}{2}\sigma^2 u''(x)\Phi(x)^2$$
$$+ \int_{y\neq 0} \Big(u(x+\Phi(x)y) - u(x) - u'(x)\Phi(x)y \cdot 1_{\{|y|<1\}} \Big) \; N(dy).$$

The limit function is continuous and vanishes at infinity. Therefore, the test functions are contained in the domain by [53] Lemma 31.7. □

Remark: In the one-dimensional case the following weaker condition is sufficient to guarantee that the test functions are contained in the domain of the solution: Let Φ be Lipschitz continuous and

$$x \longmapsto \sup_{\lambda\in]0,1[} \frac{1}{x+\lambda\Phi(x)} \in C_\infty(\mathbb{R}). \tag{2.14}$$

The products $u'\Phi$ and $u''\Phi$ are bounded for every continuous Φ, because u has compact support. The other step in the proof where we use the boundedness of Φ is in estimating $|H|$. However, (2.14) implies that for every $\widetilde{R} > 0$ there exists an $R > 0$ such that

$$|x| > R \qquad \Rightarrow \qquad |x+\lambda\Phi(x)| > \tilde{R} \qquad \text{for all } \lambda \in]0,1[\,. \tag{2.15}$$

Now let us go back to the step in the proof where we used Taylor's formula:

$$\begin{aligned}&\left|1_{\{|y|<1\}} \cdot u(X_{s-} + \Phi(X_{s-})y) - u(X_{s-}) - u'(X_{s-})\Phi(X_{s-})y\right| \\ &\quad\le \left|1_{\{|y|<1\}} \cdot y^2 \Phi(X_{s-})^2 \cdot u''(X_{s-} + \vartheta y \Phi(X_{s-}))\right|\end{aligned}$$

for a $\vartheta \in]0,1[$. Now letting $\lambda := \vartheta \cdot y$ and $\widetilde{R}$ be such that supp $u'' \subset \overline{B_{\widetilde{R}}(0)}$ we obtain by (2.15) that $\Phi(X_{s-})^2 \cdot u''(X_{s-} + \vartheta y \Phi(X_{s-}))$ is bounded.

Lemma 2.51 *Let Y^y be an $\mathbb{R}$-valued process, starting a.s. in y, which fulfills one of the following properties:*

a) It is right continuous at zero and bounded.

b) It is continuous in the mean at zero, i.e.

$$\lim_{t\downarrow 0} \mathbb{E}\, \|Y_t^y - Y_0^y\| = 0.$$

Then we have

$$\frac{1}{t}\mathbb{E} \int_0^t Y_s^y \; ds \xrightarrow[t\downarrow 0]{} y.$$

Proof: a) Consider

$$\begin{aligned}\left|\mathbb{E}\frac{1}{t}\int_0^t (Y_s^y - Y_0^y)\; ds\right| &\le \mathbb{E}\frac{1}{t}\int_0^t |Y_s^y - Y_0^y| \; ds \\ &\le \mathbb{E}\frac{1}{t} t \cdot \sup_{0\le s\le t} |Y_s^y - Y_0^y| \\ &= \mathbb{E} \sup_{0\le s\le t} |Y_s^y - Y_0^y| \\ &\xrightarrow[t\downarrow 0]{} 0.\end{aligned}$$

The result follows from Lebesgue's theorem.
b) This time we use Tonelli's theorem:

$$\begin{aligned}\left|\mathbb{E}\frac{1}{t}\int_0^t (Y_s^y - Y_0^y)\ ds\right| &\leq \frac{1}{t}\int_0^t \mathbb{E}\left|Y_s^y - Y_0^y\right|\ ds \\ &\leq \sup_{0\leq s\leq t} \mathbb{E}\left|Y_s^y - Y_0^y\right| \\ &\xrightarrow[t\downarrow 0]{} 0.\end{aligned}$$

□

Now we turn to a more general setting. We will investigate the solution of the SDE

$$X_t = x + \int_0^t \Phi(X_{s-})\ dY_s \tag{2.16}$$

where Y is an n-dimensional Hunt semimartingale. In particular the driving process does not have to be homogeneous in space any more. Let $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ be Lipschitz continuous or more generally 'acceptable' in the sense of [13].

Since Y is a universal Markov process, we have the family of probability measures $\mathbb{P}^y$ in the background. The equation (2.16) can be solved w.r.t. $\mathbb{P}^y$ for every $y \in \mathbb{R}^n$. If we want to emphasize the dependence on the starting point y we write Y^y and for the solution $X^{x,y}$.

Again the original space $(\Omega, \mathcal{F}, \mathbb{F}, Y, \mathbb{P}^y)_{y\in\mathbb{R}^n}$ might be too small to carry the solution, which now depends on x and y. An analogous construction to the one in the Lévy driven case (2.9) can be used to overcome this problem (See [49]). On this larger space we have:

Theorem 2.52 *In the above setting the vector process* $(X,Y)' = (X^{x,y}, Y^y)'$ *consisting of the solution and the driving process of* (2.16) *is a strong Markov process with transition function*

$$P_{s,s+t}^{u,w}((x,y)', B) = P_{0,t}^{x,y}((x,y)', B) = \mathbb{P}^{x,y}((X_t, Y_t)' \in B) \tag{2.17}$$

for $u, x \in \mathbb{R}^d$*,* $w, y \in \mathbb{R}^n$*,* $s, t \geq 0$ *and* $B \in \mathcal{B}^{d+n}$*. In particular we can set*

$$P_t((x,y)', B) := P_{0,t}^{x,y}((x,y)', B).$$

In order to keep the notation simple we will write (X,Y) instead of $(X,Y)'$ or $\begin{pmatrix} X \\ Y \end{pmatrix}$ for the $(d+n)$-dimensional process.

Remark: We do not give the precise definitions of some of the concepts of [13], because they are not used later on. The reader who is interested in these is advised to have a look at the original paper.

Proof: This is a corollary to Theorem (8.11) of [13] in its strong Markov version. In this particular case the process H, in the notation of [13] is zero and the driving process

Z is nothing but the underlying Markov process, in particular it is 'strongly additive' (because of the very definition of the random-shift). The coefficient Φ is 'acceptable' since it is Lipschitz continuous. Furthermore, $\Phi(X)$ is 'strongly homogeneous'. Note for the last point that (for the 'big shifts Θ')

$$\begin{aligned}(\Theta_S(\Phi(X)))_t &= (\Phi(X)_{t-S} \circ \theta_S) \cdot 1_{[[S,\infty[[}(t) \\ &= \Phi((Y_{t-S} \circ \theta_S) \cdot 1_{[[S,\infty[[}(t)) \\ &= \Phi(\Theta_S(X))_t\end{aligned}$$

holds up to an evanescent set on $[[S, \infty[[$ for every finite stopping time S. □

This result has again a converse.

Theorem 2.53 *If for each (x, y) the process $(X, Y) = (X^{x,y}, Y^y)$ from above is a time-homogeneous conservative Markov process, with a transition function not depending on $\mathbb{P}^{x,y}$ as in equation* (2.17). *Then the process Y has to be a time-homogeneous Markov process under each $\mathbb{P}^y$ (and the transition function (resp. semigroup) does not depend on y).*

Proof: Let $Q^w_{s,t}$ denote the transition function of Y, which might depend on the starting point w. Consider for $x \in \mathbb{R}^d$, $w, y \in \mathbb{R}^n$, $s, t \geq 0$ and $B \in \mathcal{B}^n$

$$\begin{aligned}Q^w_{s,s+t}(y, B) = \mathbb{P}^w(Y_{s+t} \in B | Y_s = y) &= \mathbb{P}^{0,w}(X_{s+t} \in \mathbb{R}, Y_{s+t} \in B | X_s = x, Y_s = y) \\ &= P^{0,w}_{s,s+t}((x, y)', \mathbb{R} \times B) \\ &\overset{(2.17)}{=} P^{x,y}_{0,t}((x, y)', \mathbb{R} \times B) \\ &= \mathbb{P}^{x,y}(X_t \in \mathbb{R}, Y_t \in B | X_0 = x, Y_0 = y) \\ &= \mathbb{P}^y(Y_t \in B | Y_0 = y) \\ &= Q^y_{0,t}(y, B)\end{aligned}$$

which gives us the transition semigroup $Q_t(y, B) := Q^y_{0,t}(y, B)$. In particular Y is a time-homogeneous Markov process. □

Remark: Compare in this context [38] Theorem 2.

3 Itô Processes and Feller Semimartingales

In the first section we will show that every nice Feller process is a Markov semimartingale and even an Itô process in the sense of [13]. Our Theorem 3.10 is an extension of Theorem 3.5 in [56].

In Section 2 we have a closer look at Itô processes and give some characterizations of this class of processes. In addition the reader finds a result which is quite similar to Theorem 3.10, but which is proved in a totally different way.

As before we assume that Markov processes are right continuous.

3.1 Feller Semimartingales

In the sequel X denotes a Feller process with generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and $A|_{C_c^\infty(\mathbb{R}^d)} = -p(x, D)$, i.e.

$$Au(x) = -p(x,D)u(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} p(x,\xi)\widehat{u}(\xi)\, d\xi \qquad \text{for } u \in C_c^\infty(\mathbb{R}^d)$$

where

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y)\right) N(x, dy) \tag{3.1}$$

is a negative definite symbol with Lévy triplet $(\ell(x), Q(x), N(x, dy))$ and cut-off function $\chi = \chi_R$ for an arbitrary $R \geq 1$. Remember that $1_{\overline{B_R(0)}} \leq \chi_R \leq 1_{\overline{B_{2R}(0)}}$.

Let us start with a general result giving a first connection between Feller processes and semimartingales:

Theorem 3.1 *Let X be a conservative càdlàg Feller process with generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and $A|_{C_c^\infty(\mathbb{R}^d)} = -p(x, D)$. Then X is a semimartingale with respect to every $\mathbb{P}^x$.*

Remark: Some criteria for a Feller process to be conservative can be found in Sections 1.3 and 1.4.

Proof: In this proof we follow mainly [56].
We already know that for $u \in D(A)$ the process

$$\begin{aligned} M_t := M_t^{[u]} - u(x) &= u(X_t) - \int_0^t Au(X_s)\, ds \\ &= u(X_t) + \int_0^t p(y, D)u(y)|_{y=X_s}\, ds \end{aligned}$$

is a martingale (see Theorem 1.23). Here $p(y, D)$ denotes the pseudo differential operator which has been introduced in Definition 1.11. Let $j \in \{1, .., d\}$ and $(\phi_k)_{k\in\mathbb{N}} := (\phi_k^j)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that

$$\phi_k|_{\overline{B_k(0)}} = x^{(j)}, \qquad \phi_k|_{\overline{B_{2k}(0)}^c} = 0, \qquad \|\phi_k\|_\infty \leq k + 1.$$

Then clearly we have $\phi(\cdot - x) \in C_c^\infty(\mathbb{R}^d) \subset D(A)$ and therefore

$$M_t^k := \phi_k(X_t - x) + \int_0^t p(y,D)\phi_k(y-x)|_{y=X_s}\, ds$$

are martingales for every $k \in \mathbb{N}$. Now let us have a closer look at the integral term: since $p(\cdot, D)\phi_k \in C_\infty(\mathbb{R}^d)$ this function is bounded. Therefore, we have for every partition $\pi = (0 = t_0 < t_1 < .. < t_n = t)$:

$$\begin{aligned}
&\sum_{t_j, t_{j+1} \in \pi} \left| \int_0^{t_{j+1}} p(y,D)\phi_k(y-x)|_{y=X_s}\, ds - \int_0^{t_j} p(y,D)\phi_k(y-x)|_{y=X_s}\, ds \right| \\
&= \sum_{t_j, t_{j+1} \in \pi} \left| \int_{t_j}^{t_{j+1}} p(y,D)\phi_k(y-x)|_{y=X_s}\, ds \right| \\
&\leq \sum_{t_j, t_{j+1} \in \pi} (t_{j+1} - t_j) \cdot \|p(y,D)\phi_k\|_\infty \\
&= t \cdot \|p(y,D)\phi_k\|_\infty
\end{aligned}$$

and $\int_0^t p(y,D)\phi_k(y-x)|_{y=X_s}$ is of finite variation on compacts.
This shows that

$$X_t^{(j),k} := \phi_k(X_t - x) = \underbrace{M_t^k}_{\in \mathcal{M}_{0,loc}} - \underbrace{\int_0^t p(y,D)\phi_k(y)|_{y=X_s}\, ds}_{\in \mathcal{V}} \tag{3.2}$$

is a semimartingale for every $k \in \mathbb{N}$. Set $T_k := T_k^x := \inf\{t \geq 0 : \|X_t - x\| > k\}$. Then the T_k are stopping times and since X is conservative $T_k \uparrow \infty$ a.s. Furthermore, we have

$$(X_\bullet^{(j),k})^{T_k-} = \phi_k(X_\bullet - x)^{T_k-} = (X_\bullet - x^{(j)})^{T_k-}\ . \tag{3.3}$$

This is easily seen in the case $t < T_k$:

$$\phi_k(X_t - x)^{T_k-} = \phi_k(X_t - x) = X_t^{(j)} - x^{(j)} = (X_t^{(j)} - x^{(j)})^{T_k-}$$

and if $t \geq T_k$ it follows from

$$\phi_k(X_t - x)^{T_k-} = \phi_k(X_{T_k-} - x) = \lim_{r<T_k, r\to T_k} \phi_k(X_r - x) = \lim_{r<T_k, r\to T_k} (X_r^{(j)} - x^{(j)}) = X_{T_k-}^{(j)} - x^{(j)}.$$

The equality (3.3) shows together with (3.2) that $X - x$ is pre-locally a semimartingale. Theorem 2.15 tells us that $X - x$ is a semimartingale and hence $X \in \mathcal{S}$. $\square$

Now we want to further investigate the semimartingale nature of a Feller process.

In earlier papers on this topic the following growth condition is often needed:

$$\sup_{x\in\mathbb{R}^d} |p(x,\xi)| \leq c \cdot (1 + \|\xi\|^2) \qquad \text{for } \xi \in \mathbb{R}^d. \tag{3.4}$$

We show in the following lemma that a local version of this estimate always holds for the symbols we are dealing with.

Lemma 3.2 *Let (A,D(A)) be the generator of a Feller process X such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and*

$$Au(x) = -p(x,D)u(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} p(x,\xi)\widehat{u}(\xi)\, d\xi \qquad \text{for } u \in C_c^\infty(\mathbb{R}^d).$$

In this case for every compact set $K \subset \mathbb{R}^d$ there exists a constant $c_K > 0$ such that

$$\sup_{x\in K} |p(x,\xi)| \leq c_K \cdot (1 + \|\xi\|^2) \qquad \text{for } \xi \in \mathbb{R}^d.$$

Proof: We have for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$

$$|p(x,\xi)| = \sqrt{|p(x,\xi)|}^2 = \sqrt{\left|p\left(x, k\tfrac{\xi}{k}\right)\right|}^2 \leq \left(k \cdot \sqrt{p\left(x, \tfrac{\xi}{k}\right)}\right)^2 = k^2 \left|p\left(x, \tfrac{\xi}{k}\right)\right|$$

where the inequality follows from Proposition 1.2 i). For a given ξ we choose $k_0 := \inf\{k \in \mathbb{N} : \|\xi\| \leq k\}$. In particular we have: $k_0 < \|\xi\| + 1$. Then $k_0^2 \leq (\|\xi\| + 1)^2 \leq 2(\|\xi\|^2 + 1)$ and

$$|p(x,\xi)| \leq 2(1 + \|\xi\|^2) \cdot \left|p\left(x, \frac{\xi}{k_0}\right)\right| \leq 2(1 + \|\xi\|^2) \cdot \sup_{\|\eta\|\leq 1} |p(x,\eta)|\,.$$

Let $K \subset \mathbb{R}^d$ be a compact set. We obtain

$$\begin{aligned}
\sup_{x\in K} |p(x,\xi)| &\leq \sup_{x\in K} \left(2(1 + \|\xi\|^2) \sup_{\|\eta\|\leq 1} |p(x,\eta)|\right) \\
&\leq \left(2 \sup_{x\in K} \sup_{\|\eta\|\leq 1} |p(x,\eta)|\right) \cdot (1 + \|\xi\|^2) \\
&\leq c_K \cdot (1 + \|\xi\|^2)
\end{aligned}$$

since the symbol $p(x,\xi)$ is locally bounded by Corollary 1.21. □

Next we observe that the local growth condition is equivalent to the local boundedness of the 'coefficients' ℓ, Q and $N(\cdot, dy)$:

Lemma 3.3 *Let $p(x,\xi)$ be a continuous negative definite symbol. The following two properties are equivalent:*
(a) For every compact set $K \subset \mathbb{R}^d$ there is a $c_K \in \mathbb{R}_+$ such that

$$\sup_{x\in K} |p(x,\xi)| \leq c_K \cdot (1 + \|\xi\|^2) \qquad \text{for } \xi \in \mathbb{R}^d. \tag{3.5}$$

(b) For every compact set $K \subset \mathbb{R}^d$ we have:

$$\|\ell\|_{K,\infty} + \|Q\|_{K,\infty} + \left\| \int_{y\neq 0} \left(\frac{\|y\|^2}{1+\|y\|^2}\right) \ N(\cdot, dy) \right\|_{K,\infty} < \infty \tag{3.6}$$

Proof: Analogously to [56], Lemma 2.1 □

For the proof of this lemma it is nice to use the particular truncation function $\|y\|^2/(1+\|y\|^2)$. But let us mention that this is not a cut-off function as we introduced it in Definition 1.4. One cannot use it to define the semimartingale characteristics. It turns out that in order to establish a good version of the characteristics of a Feller process one should use the same cut-off function for both: the Lévy-Khinchine representation of the symbol and the semimartingale characteristics. The following lemma allows us to switch to an arbitrary cut-off function χ.

Lemma 3.4 *We have for every compact set $K \subset \mathbb{R}^d$*

$$\left\| \int_{y\neq 0} \left(\frac{\|y\|^2}{1+\|y\|^2}\right) \ N(\cdot, dy) \right\|_{K,\infty} < \infty \Leftrightarrow \left\| \int_{y\neq 0} \left(\|y\|^2 \wedge 1\right) \ N(\cdot, dy) \right\|_{K,\infty} < \infty.$$

Proof: This follows from the simple fact that for $x \in \mathbb{R}_+$

$$\frac{x}{1+x} \le (x \wedge 1) \le 2 \cdot \frac{x}{1+x}.$$

□

Let us emphasize that, since we are dealing with a symbol which appears in the Fourier representation of the generator, the equivalent conditions of Lemma 3.3 are always met, by Lemma 3.2

Studying the domain $D(A)$ of the generator $A = -p(x, D)$ it is useful to rewrite it in the so called **integro-differential-representation**. First we need the following lemma:

Lemma 3.5 *For a cut-off function $\chi = \chi_R$, $(R \ge 1)$ as above and $y, \xi \in \mathbb{R}^d$ we have*

$$\left|e^{i\xi' y} - 1 - i\xi' y\chi(y)\right| \le 2(R+1)(1+\|\xi\|^2)(1 \wedge \|y\|^2). \tag{3.7}$$

Proof: Remember that $1_{\overline{B_R(0)}} \le \chi \le 1_{\overline{B_{2R}(0)}}$. In particular supp $u \subset \overline{B_{2R}(0)}$. Consider

$$\begin{aligned}
&\left|e^{i\xi' y} - 1 - i\xi' y\chi(y)\right| \\
&\quad \le \left|1_{\{0<\|y\|\le 1\}}(e^{i\xi' y} - 1 - i\xi' y)\right| + \left|1_{\{1<\|y\|\le 2R\}}(e^{i\xi' y} - 1 - i\xi' y\chi(y))\right| \\
&\qquad\quad + \left|1_{\{\|y\|>2R\}}(e^{i\xi' y} - 1)\right| \\
&\quad \le \left(1_{\{0<\|y\|\le 1\}}(\xi' y)^2\right) + \left(1_{\{1<\|y\|\le 2R\}}(2 + \|\xi\| \, \|y\|)\right) + \left(1_{\{\|y\|>2R\}} \cdot 2\right) \\
&\quad \le \left(1_{\{0<\|y\|\le 1\}} \|\xi\|^2 \|y\|^2\right) + \left(1_{\{1<\|y\|\le 2R\}}(2 + (1+\|\xi\|^2)2R)\right) + \left(1_{\{\|y\|>2R\}} \cdot 2R\right) \\
&\quad \le 2(R+1)(1+\|\xi\|^2)(1 \wedge \|y\|^2).
\end{aligned}$$

Where we used a Taylor expansion in the first term and the Cauchy-Schwarz inequality twice. □

For u in $C_c^\infty(\mathbb{R}^d)$ we obtain

$$\begin{aligned}
-p(x,D)u(x) &= -\int_{\mathbb{R}^d} p(x,\xi)\widehat{u}(\xi)e^{ix'\xi}\,d\xi \\
&= -\int_{\mathbb{R}^d}\Big(-i\ell(x)'\xi + \frac{1}{2}\xi'Q\xi \\
&\qquad - \int_{y\neq 0}(e^{i\xi' y} - 1 - i\xi' y\cdot\chi(y))\;N(x,dy)\Big)\,\widehat{u}(\xi)e^{ix'\xi}\,d\xi \\
&= \ell(x)'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^{n}\Big(q^{jk}(x)\partial_j\partial_k u(x)\Big) \\
&\qquad + \int_{y\neq 0}\int_{\mathbb{R}^d}\Big((e^{iy'\xi} - 1 - i\xi' y\cdot\chi(y))\widehat{u}(\xi)e^{ix'\xi}\Big)\,d\xi N(x,dy)
\end{aligned}$$

where the possibility to change the order of integration is given by the estimate (3.7). This shows that

$$-p(x,D) = I(p)|_{C_c^\infty(\mathbb{R}^d)}$$

where

$$\begin{aligned}
I(p)u(x) &:= \ell(x)'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^{d}\Big(q^{jk}(x)\partial_j\partial_k u(x)\Big) \\
&\qquad + \int_{y\neq 0}\Big(u(x+y) - u(x) - y'\nabla u(x)\cdot\chi(y)\Big)\,N(x,dy).
\end{aligned} \tag{3.8}$$

Let use remark that $I(p)$ is defined on $C_b^2(\mathbb{R}^d)$. In order to get control over the last term the following estimate is useful.

Lemma 3.6 *Let $K \subset \mathbb{R}^d$ be a compact set, $u \in C_b^2(\mathbb{R}^d)$ and $\chi = \chi_R$ the cut-off function from above. For $x \in K$ and $y \in \mathbb{R}^d$ we have*

$$\begin{aligned}
&|u(x+y) - u(x) - y'\nabla u(x)\cdot\chi(y)| \\
&\quad \le (2R)^2(\|y\|^2 \wedge 1)\left(\|u\|_\infty + \sum_{|\alpha|=1}\|\partial^\alpha u\|_{K,\infty} + \sum_{|\alpha|=2}\|\partial^\alpha u\|_{K+\overline{B_1(0)},\infty}\right).
\end{aligned} \tag{3.9}$$

In particular it follows

$$|u(x+y) - u(x) - y'\nabla u(x)\cdot\chi(y)| \le (2R)^2(\|y\|^2 \wedge 1)\sum_{|\alpha|\le 2}\|\partial^\alpha u\|_\infty . \tag{3.10}$$

Proof: Fix $x \in K$. For every $y \in \mathbb{R}^d$ we obtain:

$$\begin{aligned}
&\Big|u(x+y)-u(x)-y'\nabla u(x)\cdot\chi(y)\Big| \\
&\quad\le \Big|1_{\{0<\|y\|\le 1\}}(u(x+y)-u(x)-y'\nabla u(x)\cdot\chi(y))\Big| \\
&\qquad+\Big|1_{\{\|y\|>1\}}(u(x+y)-u(x)-y'\nabla u(x)\cdot\chi(y))\Big| \\
&\quad\le \left|1_{\{0<\|y\|\le 1\}}\left(\sum_{j,k=1}^{d}\left(\|\partial_j\partial_k u\|_{K+\overline{B_1(0)},\infty}\, y_j y_k\right)\cdot\chi(y)\right)\right| \\
&\qquad+\left|1_{\{\|y\|>1\}}\Big(2\|u\|_\infty+\|y\|\,\|\nabla u(x)\|\cdot\chi(y)\Big)\right| \\
&\quad\le 1_{\{0<\|y\|\le 1\}}\left(\sum_{j,k=1}^{d}\left(\|\partial_j\partial_k u\|_{K+\overline{B_1(0)},\infty}\right)\|y\|^2\right) \\
&\qquad+1_{\{\|y\|>1\}}\Big(2\|u\|_\infty+\|y\|^2\chi(y)\cdot|\nabla u(x)|\Big) \\
&\quad\le 1_{\{0<\|y\|\le 1\}}\|y\|^2\left(\sum_{j,k=1}^{d}\left(\|\partial_j\partial_k u\|_{K+\overline{B_1(0)},\infty}\right)\right) \\
&\qquad+1_{\{\|y\|>1\}}(2\|u\|_\infty)+1_{\{\|y\|>1\}}(2R)^2|\nabla u(x)| \\
&\quad\le (2R)^2(\|y\|^2\wedge 1)\left(\|u\|_\infty+\sum_{|\alpha|=1}\|\partial^\alpha u\|_{K,\infty}+\sum_{|\alpha|=2}\|\partial^\alpha u\|_{K+\overline{B_1(0)},\infty}\right).
\end{aligned}$$

Where we used a Taylor expansion in the first term and the Cauchy-Schwarz inequality for the Euclidean scalar product in the second one. Equation (3.10) follows directly from (3.9). □

Now we prove a local version of the inequality (2.9) of [56]. Since we have not demanded that the test functions are a core of A, there can be different extensions of it to a Feller generator, i.e. the generator of a Feller process.

Lemma 3.7 *Let (A,D(A)) be a Feller generator such that $C_c^\infty(\mathbb{R}^d)\subset D(A)$ and $A|_{C_c^\infty(\mathbb{R}^d)}=-p(x,D)$. Then $-p(x,D)$ can be extended to a Feller generator and for every such extension $(-\widetilde{p(x,D)},D(-\widetilde{p(x,D)}))$ and every compact set $K\subset\mathbb{R}^d$ there is a constant $d_K>0$ such that*

$$\left\|\widetilde{p(\cdot,D)}u\right\|_{K,\infty}\le d_K\cdot\sum_{|\alpha|\le 2}\|\partial^\alpha u\|_\infty \qquad \text{for } u\in C_b^2. \tag{3.11}$$

Proof: Since $-p(x,D)\subset A$ the operator has an extension to a Feller generator. Let $K\subset\mathbb{R}^d$ be a compact set. We may assume that $\ell=0$ and $Q=0$ as for these 'coefficients' the inequality is clear. Uniformly for $x\in K$ we obtain using formula (3.10):

$$
\begin{aligned}
\left|\widetilde{p(x,D)}u(x)\right| &= \left|\int_{y\neq 0}\Big(u(x+y)-u(x)-y'\nabla u(x)\cdot\chi(y)\Big)\ N(x,dy)\right| \\
&\leq \int_{y\neq 0}|u(x+y)-u(x)-y'\nabla u(x)\cdot\chi(y)|\ \ N(x,dy) \\
&\leq (2R)^2\cdot\underbrace{\left(\int_{y\neq 0}(\|y\|^2\wedge 1)\ N(x,dy)\right)}_{<\infty \text{ by Lemmas 3.2 - 3.4}}\cdot\sum_{|\alpha|\leq 2}\|\partial^\alpha u\|_\infty\,.
\end{aligned}
$$

Which completes the proof. □

The next observation helps us to prove that the domain of the Feller generator is quite rich: since A is the generator of a Feller semigroup it maps the test functions (which in our investigations are always in $D(A)$) into $C_\infty(\mathbb{R}^d)$. This means in particular that for $u\in C_c^\infty(\mathbb{R}^d)$ we have

$$\lim_{|x|\to\infty} Au(x)=0.$$

For $x\in$(supp $u)^c$ this reads

$$\left|\int_{y\neq 0}u(x+y)\ N(x,dy)\right| \underset{|x|\to\infty}{\longrightarrow} 0. \tag{3.12}$$

Theorem 3.8 *Let (A,D(A)) be a Feller generator such that $C_c^\infty(\mathbb{R}^d)\subset D(A)$ and $A|_{C_c^\infty(\mathbb{R}^d)}=-p(x,D)$. Then $-p(x,D)$ can be extended to a Feller generator and for every such extension $(-\widetilde{p(x,D)},D(-\widetilde{p(x,D)}))$ we have*

$$C_c^2(\mathbb{R}^d)\subset D(-\widetilde{p(x,D)}).$$

Proof: The operator $-\widetilde{p(x,D)}$ is closed. Let $u\in C_c^2(\mathbb{R}^d)$. By Corollary C.2 of the appendix we know that there is a sequence $(u_n)_{n\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \xrightarrow[n\to\infty]{\sum_{|\alpha|\leq 2}\|\partial^\alpha\cdot\|_\infty} u \qquad\text{and}\qquad \text{supp } u_n\subset K \text{ (compact) } \forall n\in\mathbb{N}.$$

If we could show that the sequence $(-\widetilde{p(x,D)}u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C_\infty(\mathbb{R}^d),\|\cdot\|_\infty)$ the assertion would follow because of the closedness of the operator. Let $\varepsilon>0$. For $x\in K^c$ we have

$$-p(x,D)(u_m-u_n)(x)=\int_{y\neq 0}(u_m-u_n)(x+y)\ N(x,dy).$$

Since the convergence of $(u_n)_{n\in\mathbb{N}}$ is uniform, the sequence $(u_n)_{n\in\mathbb{N}}$ has to be uniformly bounded (and the support of every u_n is in K). Therefore, we can find a function $f\in C_c^\infty(\mathbb{R}^d)$ such that $-f\leq u_n\leq f$ for every $n\in\mathbb{N}$. This implies

$$-2f\leq(u_m-u_n)\leq 2f,\ \forall n,m\in\mathbb{N}$$

and we obtain using (3.12)

$$\underbrace{-\int_{y\neq 0}2f(x+y)\ N(x,dy)}_{\underset{|x|\to\infty}{\longrightarrow}0}\leq\int_{y\neq 0}(u_m-u_n)(x+y)\ N(x,dy)\leq\underbrace{\int_{y\neq 0}2f(x+y)\ N(x,dy)}_{\underset{|x|\to\infty}{\longrightarrow}0}$$

which shows that the integral in the middle tends to 0 uniformly in n, m. This means that there exists a compact set $\tilde{K} \supset K$ such that for $x \in \tilde{K}^c$ we obtain

$$|p(x, D)(u_m - u_n)(x)| = \left| \int_{y \neq 0} (u_m - u_n)(x + y) \ N(x, dy) \right| < \varepsilon. \tag{3.13}$$

For $x \in \tilde{K}$ we use formula (3.11):

$$\|p(\cdot, D)(u_m - u_n)\|_{\tilde{K},\infty} \leq d_{\tilde{K}} \cdot \sum_{|\alpha| \leq k} \|\partial^\alpha (u_m - u_n)\|_\infty$$

since the u_n converge in the norm $\sum_{|\alpha|\leq 2} \|\partial^\alpha \cdot\|_\infty$ we can find an $N \in \mathbb{N}$ such that for every $n, m \geq N$

$$\|p(\cdot, D)(u_m - u_n)\|_{\tilde{K},\infty} < \varepsilon.$$

Together with formula (3.13) this yields the asserted Cauchy property. □

Corollary 3.9 *Under the assumptions of Theorem 3.8 every extension* $(\widetilde{-p(x, D)}, D(\widetilde{-p(x, D)}))$ *fulfills*

$$\widetilde{-p(x, D)}|_{C_c^2(\mathbb{R}^d)} = I(p)$$

where $I(p)$ *is given by* (3.8).

Proof: On $C_c^\infty(\mathbb{R}^d)$ the operators are the same and the image-sequence $(\widetilde{-p(x, D)}u_n)_{n\in\mathbb{N}}$ in the proof above converges uniformly in $C_\infty(\mathbb{R}^d)$. □

Now we can prove the main result of this section.

Theorem 3.10 *Let* $(A, D(A))$ *be a Feller generator such that* $C_c^\infty(\mathbb{R}^d) \in D(A)$ *and* $A|_{C_c^\infty(\mathbb{R}^d)} = -p(x, D)$ *with symbol, see* (3.1),

$$p(x, \xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y) \right) \ N(x, dy).$$

Let X be the Feller process, generated by any extension $\widetilde{-p(x, D)}$ to a Feller generator. Under the assumptions that X is conservative, this process is an Itô process and its semimartingale characteristics (B, C, ν) with respect to χ are

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega)) \ ds \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega)) \ ds \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy) \ ds \end{aligned} \tag{3.14}$$

for every $\mathbb{P}^x, (x \in \mathbb{R}^d)$ where $(\ell, Q, N(\cdot, dy))$ are the Lévy characteristics which appear in the symbol of the Feller process.

Proof: We already know that X is a semimartingale. By Theorem 2.26 it suffices to show that for every $u \in C_b^2(\mathbb{R}^d)$ and every $\mathbb{P}^x$ ($x \in \mathbb{R}^d$) the process given by

$$\begin{aligned}
\widetilde{M}_t^{[u]} &= u_n(X_t) - u_n(X_0) - \sum_{j=1}^d \int_0^t \Big(\partial_j u_n(X_{s-})\ell^{(j)}(X_{s-})\Big)\, ds \\
&-\frac{1}{2}\sum_{j,k=1}^d \int_0^t \Big(\partial_j\partial_k u_n(X_{s-})q^{jk}(X_{s-})\Big)\, ds \\
&-\int_0^t \int_{y\neq 0} \Big(u_n(X_{s-}+y) - u_n(X_{s-}) - \chi(y)y'\nabla u_n(X_{s-})\Big)\, N(X_{s-},dy)\, ds
\end{aligned}$$

is a local martingale. Fix $u \in C_b^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Now let $\tilde{\chi} \in C_c^\infty$ be a (smooth) cut-off function and let $\chi_n := \tilde{\chi}(\cdot/n)$ for $n \in \mathbb{N}$. For the sequence $(u_n)_{n\geq 0}$ defined by $u_n := u \cdot \chi_n$ we obtain

$$\sum_{|\alpha|\leq 2} \|\partial^\alpha u_n\|_\infty \leq c < \infty \text{ uniformly in } n \in \mathbb{N} \tag{3.15}$$

by Leibniz's rule for differentiation. Furthermore, we have by definition $u_n \in C_c^2(\mathbb{R}^d) \subset D(-\widetilde{p(x,D)})$ and $u_n \to u$, $\partial_j u_n \to \partial_j u$ and $\partial_j\partial_k u_n \to \partial_j\partial_k u$ for $j,k = 1,...,d$, where the convergence is locally uniform. By Theorem 1.22 we conclude that

$$\begin{aligned}
M_t^{[u_n]} &= u_n(X_t) - u_n(X_0) + \int_0^t \widetilde{p(x,D)}u_n(X_s)\, ds \\
&= u_n(X_t) - u_n(X_0) + \int_0^t \widetilde{p(x,D)}u_n(X_{s-})\, ds
\end{aligned}$$

is a $\mathbb{P}^x$-martingale. Using the explicit representation of $I(p)$ (see (3.8)) we obtain by Corollary 3.9

$$M_t^{[u_n]} = \widetilde{M}_t^{[u_n]}.$$

Let $T_K := T_K^x := \inf\{t \geq 0 : \|X_t - x\| > K\}$ for every $K \in \mathbb{N}$. Then $(T_K)_{K\in\mathbb{N}}$ is a sequence of stopping times such that $T_K \uparrow \infty$ a.s. by the conservativeness of the process. The stopped processes $(\widetilde{M}_t^{[u_n]})^{T_K}$ are martingales for every $n, K \in \mathbb{N}$ by (0.36). Let us have a closer look at these processes:

$$\begin{aligned}
(\widetilde{M}_t^{[u_n]})^{T_K} &= u_n(X_t^{T_K}) - u_n(X_0^{T_K}) - \int_0^{t\wedge T_K} I(p)u_n(X_{s-})\, ds \\
&= u_n(X_t^{T_K}) - u_n(X_0^{T_K}) - \int_0^t I(p)u_n(X_{s-})1_{[[0,T_K]]}(\cdot,s)\, ds.
\end{aligned}$$

Using again the explicit representation of $I(p)$ we can write:

$$
\begin{aligned}
(\widetilde{M}_t^{[u_n]})^{T_K} &= u_n(X_t^{T_K}) - u_n(X_0^{T_K}) \\
&- \sum_{j=1}^d \int_0^t \Big(\partial_j u_n(X_{s-})\ell^{(j)}(X_{s-})1_{[[0,T_K]]}(\cdot,s)\Big)\,ds \\
&- \frac{1}{2}\sum_{j,k=1}^d \int_0^t \Big(\partial_j\partial_k u_n(X_{s-})q^{jk}(X_{s-})1_{[[0,T_K]]}(\cdot,s)\Big)\,ds \qquad (3.16)\\
&- \int_0^t \int_{y\neq 0} \Big((u_n(X_{s-}+y) - u_n(X_{s-}) - \chi(y)y'\nabla u_n(X_{s-}))\times \\
&\qquad 1_{[[0,T_K]]}(\cdot,s)\Big)\,N(X_{s-},dy)\,ds.
\end{aligned}
$$

Since $(\widetilde{M}_t^{[u_n]})_t^{T_K}$ is a martingale for every $n, K \in \mathbb{N}$ we obtain for $r \le t$ and $F \in \mathcal{F}_r$

$$\int_F (\widetilde{M}_t^{[u_n]})^{T_K}\,d\mathbb{P}^x = \int_F (\widetilde{M}_r^{[u_n]})^{T_K}\,d\mathbb{P}^x.$$

If we show that for every $r \le t$ and $F \in \mathcal{F}_r$ (the case $r = t$ is included)

$$\int_F (\widetilde{M}_t^{[u_n]})^{T_K}\,d\mathbb{P}^x \xrightarrow[n\to\infty]{} \int_F (\widetilde{M}_t^{[u]})^{T_K}\,d\mathbb{P}^x \qquad (3.17)$$

we would obtain that $(\widetilde{M}_t^{[u]})_{t\geq 0}^{T_K}$ is a martingale which would yield in turn that $(\widetilde{M}_t^{[u]})_{t\geq 0}$ is a local martingale and hence the result.
Therefore, the only thing which remains to be proved is (3.17). We fix $K \in \mathbb{N}$, $r \le t$ and $F \in \mathcal{F}_r$ and show the convergence separately for every term in (3.16):
We start with term number one: since $\|\chi_n\|_\infty \le 1$ and u is bounded the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded. Furthermore, we have that $u_n(X_t^{T_K})$ converges pointwise to $u(X_t^{T_K})$ and by Lebesgue's theorem we obtain

$$\int_F u_n(X_t^{T_K})\,d\mathbb{P}^x \xrightarrow[n\to\infty]{} \int_F u(X_t^{T_K})\,d\mathbb{P}^x.$$

The second term, $u_n(X_0^{T_K})$, works alike.
In term three we obtain (for $j = 1, ..., d$) that $\ell(X_{s-}) \cdot 1_{[[0,T_K]]}(\cdot, s)$ is bounded because ℓ is locally bounded (see Lemma 3.3) and $\|X_{s-}\| \le K$ on $[[0, T_K]]$. Furthermore, $\|\partial_j u_n\|_\infty$ is bounded in n (see (3.15)). Using Lebesgue's theorem on the space $F \times [0, t]$ we obtain

$$
\begin{aligned}
&\int_F \int_0^t \partial_j u_n(X_{s-})\ell^{(j)}(X_{s-})1_{[[0,T_K]]}(\cdot,s)\,ds\,d\mathbb{P}^x \\
&\xrightarrow[n\to\infty]{} \int_F \int_0^t \partial_j u(X_{s-})\ell^{(j)}(X_{s-})1_{[[0,T_K]]}(\cdot,s)\,ds\,d\mathbb{P}^x.
\end{aligned}
$$

The fourth term works like the third one. The only difference is that now second derivatives are used.
In term five the pointwise convergence of the integrand

$$u_n(X_{s-}(\omega)+y) - u_n(X_{s-}(\omega)) - \chi(y)y'\nabla u_n(X_{s-}(\omega)) \cdot 1_{[[0,T_K]]}(\omega,s)$$

for fixed (ω, s, y) is clear. The possibility to use Lebesgue's theorem is this time given by formula (3.10) and the estimate (3.15):

$$\begin{aligned}|u_n(X_{s-}+y)-u_n(X_{s-})-\chi(y)y'\nabla u_n(X_{s-})| &\le (\|y\|^2\wedge 1)\cdot \sum_{|\alpha|\le 2}\|\partial^\alpha u_n\|_\infty (2R)^2\\ &\le (\|y\|^2\wedge 1)\cdot c\cdot (2R)^2.\end{aligned}$$

We have thus established (3.17) and the result follows. □

3.2 More on Itô Processes

In this section we collect some interesting facts on Itô processes and prove a theorem which is closely linked to Theorem 3.10 but deals with the extended generator. The comments on filtrations and on the setting of [13] we made at the end of Section 1.2 are still in place.

Within this section

$$\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathbb{F} = (\mathcal{F}_t)_{t\ge 0}, (\vartheta_t)_{t\ge 0}, (X_t)_{t\ge 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d} \tag{3.18}$$

is a Hunt semimartingale, i.e. it is a quasi left continuous strong Markov process which is a semimartingale with respect to $\mathbb{P}^x$ for every $x \in \mathbb{R}^d$.[7] Therefore, $U := X - X_0$ is a semimartingale additive functional. In this situation the following proposition holds:

Proposition 3.11 *Let U be a d-dimensional conservative Markov semimartingale additive functional on the space $\mathbf{X}$ (see (3.18)). There exist*

(i) an $F \in \mathcal{V}^+_{ad}$, which admits a predictable version;

(ii) an optional process $\tilde{b} = (\tilde{b}^{(j)})_{1\le j\le d}$;

(iii) an optional process $\tilde{c} = (\tilde{c}^{jk})_{1\le j,k\le d}$ with values in the symmetric nonnegative matrices;

(iv) a transition kernel $\tilde{K}(\omega, s; dy)$ from $(\Omega\times\mathbb{R}_+, \mathcal{O})$ into $(\mathbb{R}^d\backslash\{0\}, \mathcal{B}(\mathbb{R}^d\backslash\{0\}))$, satisfying $\int (\|y\|^2\wedge 1)\ \tilde{K}(dy) < \infty$

such that

$$B = \tilde{b}\bullet F,\ \ C = \tilde{c}\bullet F,\ \ \nu(\omega; ds, dy) = \tilde{K}(\omega, s; dy)\ dF_s \tag{3.19}$$

form a version of the local characteristics of U with respect to a given cut-off function χ.

Proof: This proposition is proved in [13] (Theorem (6.25)), but only for the cut-off function $\tilde{\chi} := 1_{\overline{B_1(0)}}$. However, one can switch to an arbitrary cut-off function χ by using Proposition II.2.24 of [36]: Let $B(\chi)$ (resp. $(B(\tilde{\chi}))$ denote the second characteristic with respect to χ (resp. $\tilde{\chi}$). We obtain

$$\begin{aligned}B(\chi)_t &= B(\tilde{\chi})_t + \int_{[0,t]\times(\mathbb{R}^d\backslash\{0\})} \Big(\chi(y)y - \tilde{\chi}(y)y\Big)\ \nu(\cdot; ds, dy)\\ &= \int_0^t \tilde{b}_s\ dF_s + \int_0^t\int_{y\ne 0}\Big(\chi(y)y - \tilde{\chi}(y)y\Big)\tilde{K}(\omega, s; dy)\ dF_s\\ &= \int \left(\tilde{b}_s + \int_0^t\int_{y\ne 0}\Big(\chi(y)y-\tilde{\chi}(y)y\Big)\tilde{K}(\omega, s; dy)\right)\ dF_s.\end{aligned}$$

[7] In this definition we follow [13], i.e. we assume neither normality nor Borel measurability

Since the integrand $(\chi(y)y - \tilde{\chi}(y)y)$ is bounded, and zero in a neighborhood of zero, and since the other characteristics do not depend on the choice of the cut-off function we obtain the desired representation of the characteristics. □

The Itô processes we introduced in Section 2.2 form a subclass of Hunt semimartingales. There are some nice characterizations of Itô processes within this class. To obtain these we need the extended generator (see Definition 1.17) and the following two technical notions:

Definition 3.12 (i) A class Γ of functions is said to be a **full class** if for all $R \in \mathbb{N}$ and $j \in \{1, ..., d\}$ there exists a finite family $\{f_1, .., f_d\} \subset \Gamma$ and $g \in C_b^2(\mathbb{R}^d)$ such that $x^{(j)} = g(f_1(x), .., f_d(x))$ for all $x \in \mathbb{R}^d$, $\|x\| \leq R$.

(ii) A class of Borel functions on $\mathbb{R}^d$ is said to be a **complete class** if it contains a countable subset $\Gamma \subset C_b^2(\mathbb{R}^d)$ with the property that, for every $x \in \mathbb{R}^d$ the countable collection of numbers

$$\sum_{j=1}^d \beta^{(j)}\partial_j f(x) + \frac{1}{2}\sum_{j,k=1}^d \gamma^{jk}\partial_j\partial_k f(x) + \int \left(f(x+y) - f(x) - 1_{\overline{B_1(0)}}(y)\sum_{j=1}^d y^{(j)}\partial_j f(x)\right) \rho(dy)$$

$f \in \Gamma$ completely determines the vector $\beta \in \mathbb{R}^d$, the symmetric nonnegative matrix γ and the Lévy measure ρ.

Let us remark that every set containing the test functions is full and complete.

Theorem 3.13 *Let X be a Hunt semimartingale. The following statements are equivalent:*

(i) X is an Itô process (see Definition 2.25), i.e. it has characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\,ds \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega))\,ds \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy)\,ds \end{aligned}$$

with respect to χ.

(ii) The domain $D(A_{ext})$ of its extended generator is a full and complete class.

(iii) In the representation of Proposition 3.11 one has $F_t = t$.

In this case we have $C_b^2(\mathbb{R}^d) \subset D(A_{ext})$, and with the triplet (ℓ, Q, N) of (i) the operator $I(p)$, which was defined in (3.8) *is a version of the restriction of A_{ext} to $C_b^2(\mathbb{R}^d)$.*

The interesting fact in this theorem is that due to the Markov property the weaker appearing condition (iii) implies (i).

Proof: For the cut-off function $\tilde{\chi} := 1_{\overline{B_1(0)}}$ we have the following implications:

$(ii) \Leftrightarrow (iii)$: Theorem (7.16) (i) in [13].

$(iii) \Rightarrow (i)$: Theorem (7.14) (iii) in [13] with $F_t = t$. Note that adding the deterministic starting point does not change the characteristics.

$(i) \Rightarrow (iii)$: trivial.

In this case the operator $I(p)$ reads

$$\begin{aligned} I(p)u(x) &= \ell(x)'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^{d}\Big(q^{jk}(x)\partial_j\partial_k u(x)\Big) \\ &\quad + \int_{y\neq 0}\Big(u(x+y) - u(x) - y'\nabla u(x)\cdot 1_{\overline{B_1(0)}}(y)\Big)\, N(x,dy) \end{aligned} \tag{3.20}$$

and the last statement follows from Theorem (7.16) (ii) in [13].

It is possible to switch to an arbitrary cut-off function χ, as we did in the proof of Proposition 3.11. If we write $\ell_\chi(\cdot)^{(j)}$ to emphasize the dependence on χ, we just have to observe, that

$$\ell_{\tilde{\chi}}(\cdot)^{(j)} = \ell_\chi(\cdot)^{(j)} + \int_{y\neq 0}\Big(\tilde{\chi}(y)y - \chi(y)y\Big)\, N(\cdot, dy)$$

and plug this into formula (3.20). □

We will deal with Itô processes which have (finely) continuous differential characteristics.

Remarks: a) The class of Itô processes can, in principle, be obtained as solutions of general SDEs which are of the form

$$\begin{aligned} X_t &= x + \int_0^t \ell(X_s)\, ds + \int_0^t \sigma(X_s)\, d\tilde{W}_s \\ &+ \int_0^t\int_{\mathbb{R}} k(X_{s-}, z)1_{\{|k(X_{s-},z)|\leq 1\}}\, \Big(\tilde{\mu}(ds,dz) - ds\; N(dz)\Big) \\ &+ \int_0^t\int_{\mathbb{R}} k(X_{s-}, z)1_{\{|k(X_{s-},z)|> 1\}}\, \tilde{\mu}(ds,dz). \end{aligned} \tag{3.21}$$

where $\tilde{W}_s$ and $\tilde{\mu}$ are a Brownian motion respectively a Poisson random measure on a suitable probability space, the so called Markov extension of the original probability space (see [12]). We will deal with this concept in Section 5.3. In fact this one-to-one correspondence justifies the name 'Itô process', because K. Itô introduced this class in [27], at least under some additional conditions which guarantee existence and uniqueness of the solution.

b) It is an interesting question, for which ℓ, Q, N in Definition 2.25 or ℓ, σ, k (above) there exists an Itô process. Our generalized symbol connects this question with the question if there exists a Feller (or Itô) process with a given symbol. Some sufficient results are known in the case of the SDE, e.g. Lipschitz conditions (see [43] Section 3). In the case of symbols we refer the reader to the references we mentioned in Section 1.4. There is a close link between the coefficients of the SDE and the characteristic triplet (see Section 5.3).

Some rather technical results of [13] can be used in order to obtain a result which is quite similar to Theorem 3.10.

Theorem 3.14 *Let X be a càdlàg conservative Feller process with generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and $A|_{C_c^\infty(\mathbb{R}^d)} = -p(x, D)$ with symbol*

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y \cdot \chi(y)\right) N(x, dy).$$

Then X is an Itô process with semimartingale characteristics (B, C, ν)

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\, ds \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega))\, ds \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy)\, ds \end{aligned} \tag{3.22}$$

with respect to χ. Furthermore, we have $C_b^2(\mathbb{R}^d) \subset D(A_{ext})$, and the operator $I(p)$ defined in (3.8) *is a version of the restriction of A_{ext} to $C_b^2(\mathbb{R}^d)$.*

Proof: Every Feller process X is strongly Markovian. Furthermore, every set containing the test functions is full and complete and the domain of the generator is contained in the domain of the extended generator. By Theorem 3.13 X is an Itô process. The second part of this theorem yields that $C_b^2(\mathbb{R}^d) \subset D(A_{ext})$, where the latter denotes the domain of the extended generator, and on these functions the extended generator A_{ext} reads as follows:

$$\begin{aligned} I(p)u(x) &= \widetilde{\ell}(x)'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^{d} \left(\widetilde{q}^{jk}(x)\partial_j\partial_k u(x)\right) \\ &+ \int_{y\neq 0} \left(u(x+y) - u(x) - y'\nabla u(x)\cdot\chi(y)\right) \widetilde{N}(x, dy). \end{aligned}$$

Comparing this to the original generator on the test functions we obtain $\widetilde{\ell} = \ell$, $\widetilde{Q} = Q$ and $\widetilde{N} = N$ again by the completeness of the domain of the extended generator. □

Remarks: a) In fact it is enough if the domain of the extended generator is a full and complete class. However, it might be very difficult to check this in a given situation.

b) The above theorem says nothing about the domain of the generator of the process. Compare, however, Theorem 3.10.

c) Every Hunt semimartingale can be obtained from an Itô process as a ‘random time change’ (see [12] Theorem (3.35)).

4 The Symbol of a Stochastic Process

In this chapter we introduce the (stochastic) symbol of a universal Markov process. It turns out that in the cases where we already have an (analytic) symbol both concepts coincide. A posteriori this justifies the name.

4.1 Definition and First Example

We now introduce the central object of our investigations.

Definition 4.1 Let X be a $\mathbb{R}^d$-valued (universal) Markov process, which is conservative and normal. For $x, \xi \in \mathbb{R}^d$ we call

$$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t-x)'\xi}-1}{t} \tag{4.1}$$

the **symbol** of the process, if the limit exists.

We will see in the following that this probabilistic symbol is a generalization of the symbol we encountered in the Fourier representation of a Feller process. In the case of Lévy processes the calculation of the symbol is elementary.

Example: Let Z be a Lévy process:

$$\begin{aligned}
-\lim_{t\downarrow 0} \mathbb{E}^z \frac{e^{i(Z_t-z)'\xi}-1}{t} &= -\lim_{t\downarrow 0} \frac{\mathbb{E}^z\big(e^{i(Z_t-z)'\xi}\big)-1}{t} \\
&= -\lim_{t\downarrow 0} \frac{\mathbb{E}^0\big(e^{iZ_t'\xi}\big)-1}{t} \\
&= -\lim_{t\downarrow 0} \frac{e^{-t\psi(\xi)}-1}{t} \\
&= -\frac{\partial^+}{\partial t}\bigg|_{t=0} \Big(e^{-t\psi(\xi)}\Big) \\
&= \psi(\xi).
\end{aligned}$$

The example shows that in the case of Lévy processes our probabilistic symbol is the characteristic exponent (see Theorem 1.25) as well as the symbol of the generator (see Theorem 1.27). For Feller processes which fulfill condition (3.4), it was shown in [54] that the limit in (4.1) exists and coincides with the symbol of the generator of the process. In the following section we calculate the symbol of an Itô process. This turns out to be something similar to Feller processes.

4.2 The Symbol of an Itô Process

Recall that an Itô process is a Markov semimartingale with characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\ ds \quad j=1,...,d \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega))\ ds \quad j,k=1,...,d \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy)\ ds \end{aligned} \tag{4.2}$$

with respect to a fixed cut-off function χ and that we call the triplet $(\ell, Q, N(\cdot, dy))$ the differential characteristics of the process (see Definition 2.25).

In the following theorem the notion of fine continuity appears. This is a weaker condition than (ordinary) continuity. For more details on this concept we refer the reader to Appendix A. The most important fact is that if $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is nearly Borel measurable and finely continuous (with respect to X) then the mapping $t \longmapsto f(X_t)$ is right continuous almost surely.

Theorem 4.2 *Let X be an Itô process and let $\ell = (\ell^{(j)})_{1\leq j\leq d}$ and $Q = (q^{jk})_{1\leq j,k\leq d}$ be finely continuous and bounded; N be such that the function*

$$x \longmapsto \int_{y\neq 0} (1\wedge y^2)\ N(x, dy)$$

is finely continuous and bounded. In this case the limit (4.1) *exists and the symbol of X is*

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy'\xi\cdot\chi(y)\right)\ N(x,dy).$$

Proof: First we use Itô's formula under the expectation and obtain

$$\frac{1}{t}\mathbb{E}^x\left(e^{i(X_t-x)'\xi} - 1\right) = \frac{1}{t}\mathbb{E}^x\left(\underbrace{\int_{0+}^t i\xi e^{i(X_{s-}-x)'\xi} dX_s}_{:=I_1} + \underbrace{\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}-x)'\xi} d[X,X]_s^c}_{:=I_2} \right.$$
$$\left. + \underbrace{e^{-ix'\xi}\sum_{0<s\leq t}\left(e^{iX_s'\xi} - e^{iX_{s-}'\xi} - i\xi e^{i\xi' X_{s-}}\Delta X_s\right)}_{:=I_3}\right).$$

In what follows we will deal with the terms one-by-one in the one-dimensional case. In the multidimensional setting the notation becomes more involved, but the calculations work alike.
To calculate the first term we use the canonical decomposition of the semimartingale (see [36] Theorem II.2.34):

$$X_t = X_0 + X_t^c + \int_0^t \chi(y)y\ \left(\mu^X(\cdot; ds, dy) - \nu(\cdot; ds, dy)\right) + \check{X}(\chi) + B_t(\chi).$$

We use the linearity of the stochastic integral. Our first step is to prove:

$$\mathbb{E}^x \int_{0+}^t i\xi e^{i(X_{s-}-x)'\xi} \, dX_s^c = 0.$$

The integral $e^{i(X_{t-}-x)'\xi} \bullet X_t^c$ is a local martingale, since X_t^c is a local martingale. To see that it is indeed a martingale, we calculate the following:

$$\begin{aligned}
[e^{i(X_{t-}-x)'\xi} \bullet X^c, e^{i(X_{t-}-x)'\xi} \bullet X^c]_t &= (e^{i(X_{t-}-x)'\xi})^2 \bullet \langle X^c, X^c \rangle_t \\
&= (e^{i(X_{t-}-x)'\xi})^2 \bullet C_t \\
&= \int_0^t (e^{i(X_{s-}-x)'\xi})^2 Q(X_s(\omega)) \, ds \\
&\leq \|Q\|_\infty \cdot t < \infty.
\end{aligned}$$

This term is finite, because Q is bounded. Theorem 2.41 tells us that $e^{i(X_{t-}-x)'\xi} \bullet X_t^c$ is an L^2 martingale, which is zero at time 0. Hence its expectation is constantly zero. The same is true for the second integrand. We show that the function $H_{x,\xi}(\omega, s, y) := e^{i(X_{s-}-x)'\xi} \cdot y\chi(y)$ is in the class F_p^2 of Ikeda and Watanabe (see Definition 2.20 (v)), i.e.

$$\mathbb{E}^x \int_0^t \int_{y \neq 0} \left| e^{i(X_{s-}-x)'\xi} \cdot y\chi(y) \right|^2 \nu(\cdot; ds, dy) < \infty.$$

To prove this we observe

$$\mathbb{E}^x \int_0^t \int_{y \neq 0} \left| e^{i(X_{s-}-x)'\xi} \right|^2 \cdot |y\chi(y)|^2 \, \nu(\cdot; ds, dy) = \mathbb{E}^x \int_0^t \int_{y \neq 0} |y\chi(y)|^2 \, N(X_s, dy) \, ds.$$

Since we have by hypothesis $\left\| \int_{y \neq 0} (1 \wedge y^2) \; N(\cdot, dy) \right\|_\infty < \infty$ this expected value is finite. Therefore, the function $H_{x,\xi}$ is in F_p^2 and we conclude by Theorem 2.21 b) that

$$\begin{aligned}
&\int_0^t e^{i(X_{s-}-x)'\xi} \, d\left(\int_0^s \int_{y \neq 0} \chi(y) y \; (\mu^X(\cdot; dr, dy) - \nu(\cdot; dr, dy)) \right) \\
&\qquad = \int_0^t \int_{y \neq 0} \left(e^{i(X_{s-}-x)'\xi} \chi(y) y \right) (\mu^X(\cdot; ds, dy) - \nu(\cdot; ds, dy))
\end{aligned}$$

is a martingale. The last equality follows from [36] Theorem I.1.30.
Next we obtain

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}^x \int_{0+}^t i\xi e^{i(X_{s-}-x)'\xi} dB_s = i\xi \ell(x)$$

which is already the first term of the symbol. To prove this we use the 'associativity' of the stochastic integral (Proposition 2.31 h):

$$\begin{aligned}
\int_{0+}^t i\xi e^{i(X_{s-}-x)'\xi} dB_s &= i\xi \int_{0+}^t e^{i(X_{s-}-x)'\xi} \ell(X_{s-}) \, ds \\
&= i\xi \int_0^t e^{i(X_s-x)'\xi} \ell(X_s) \, ds.
\end{aligned}$$

For the last equality we used the fact that we integrate with respect to Lebesgue measure (and a càdlàg path has only a countable number of jumps). Since ℓ is finely continuous we obtain using Theorem A.3

$$\lim_{t\downarrow 0} i\xi \frac{1}{t}\mathbb{E}^x \int_0^t e^{i(X_s-x)'\xi}\ell(X_s)\ ds = i\xi\ell(x)$$

by Lemma 2.51 a). The other process has to be compared with the jump part (I_3). For this purpose we write it in a slightly different form:

$$\begin{aligned}
&\int_{0+}^t i\xi e^{i(X_{s-}-x)'\xi}\ d\left(\int_0^s \int_{y\neq 0} (y-y\chi(y))\ \mu^X(\cdot; dr, dy)\right)\\
&\quad= \int_{0+}^t \int_{y\neq 0} \left(i\xi e^{i(X_{s-}-x)'\xi}(y-y\chi(y))\right)\ \mu^X(\cdot; ds, dy)\\
&\quad= \int_{]0,t]\times\{y\neq 0\}} \left(e^{i(X_{s-}-x)'\xi}(i\xi' y\cdot(1-\chi(y)))\right)\ \mu^X(\cdot; ds, dy).
\end{aligned}$$

Since $\chi(y)=1$ near the origin we have to deal only with a finite number of jumps. In the second term the calculation is very similar to the integral with respect to $(B_t)_{t\geq 0}$ in the first term. The only differences are that in the multidimensional case one has to deal with a matrix instead of a vector and that the second characteristic is used instead of the first. One obtains

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^x \frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}-x)'\xi}\ d[X,X]_s^c = -\frac{1}{2}\xi^2 Q(x).$$

Now we have to deal with the jump part. at first we write the sum as an integral with respect to the jump measure μ^X of the process:

$$\begin{aligned}
&e^{-ix'\xi}\sum_{0<s\leq t}\left(e^{iX_s'\xi}-e^{iX_{s-}'\xi}-i\xi e^{i\xi X_{s-}}\Delta X_s\right)\\
&\quad= e^{-ix'\xi}\sum_{0<s\leq t}\left(e^{iX_{s-}'\xi}(e^{i\xi'\Delta X_s}-1-i\xi'\Delta X_s)\right)\\
&\quad= \int_{]0,t]\times\mathbb{R}^d}\left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y}-1-i\xi' y)1_{\{y\neq 0\}}\right)\ \mu^X(\cdot; ds, dy)\\
&\quad= \int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y}-1-i\xi' y\chi(y)-i\xi' y\cdot(1-\chi(y)))\right)\ \mu^X(\cdot; ds, dy)\\
&\quad= \int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y}-1-i\xi' y\chi(y))\right)\ \mu^X(\cdot; ds, dy)\\
&\qquad+ \int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)'\xi}(-i\xi' y\cdot(1-\chi(y)))\right)\ \mu^X(\cdot; ds, dy).
\end{aligned}$$

The last term cancels with the one we left behind from I_1. For the remainder-term we get:

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x \int_{]0,t]\times\{y\neq 0\}} \left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y} - 1 - i\xi' y\chi(y))\right) \mu^X(\cdot; ds, dy) \\
&= \frac{1}{t}\mathbb{E}^x \int_{]0,t]\times\{y\neq 0\}} \left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y} - 1 - i\xi' y\chi(y))\right) \nu(\cdot; ds, dy) \\
&= \frac{1}{t}\mathbb{E}^x \int_{0+}^t \underbrace{e^{i(X_{s-}-x)'\xi} \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y\chi(y)\right) N(X_{s-}, dy)}_{:=g(s-,\omega)}\ ds \\
&= \frac{1}{t}\mathbb{E}^x \int_0^t \left(e^{i(X_s-x)'\xi} \int_{y\neq 0} \left(e^{i\xi' y} - 1 - i\xi' y\chi(y)\right) N(X_s, dy)\right)\ ds.
\end{aligned}$$

Here we used the fact that it is possible to integrate with respect to the compensator of a random measure instead of the measure itself, if the integrand is in F_p^1 (see Theorem 2.21 a)). The function $g(s,\omega)$ is measurable and bounded by our assumption, since (3.7) yields that $\left|e^{i\xi' y} - 1 - i\xi' y\chi(y)\right| \le const \cdot (1 \wedge \|y\|^2)$. Hence $g \in F_p^1$.
Using Lemma 2.51 a) we obtain

$$\begin{aligned}
&\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^x \int_0^t e^{i(X_s-x)'\xi} \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy'\xi\chi(y)\right) N(X_s, dy)\ ds \\
&\qquad = \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy\xi\chi(y)\right) N(x, dy).
\end{aligned}$$

This is the last part of the symbol. Here we used the continuity assumption on $N(x, dy)$.

□

Remark: If every part of the triplet $(\ell(x), Q(x), \int_{y\neq 0}(1 \wedge y^2)\ N(x, dy))$ is continuous as a function in x, this implies the continuity of the symbol in x, but not the other way around. For $\alpha(x) = 1 + e^{-x^2/2}$ the negative definite symbol $|\xi|^{\alpha(x)}$ is (bi-)continuous in (x, ξ), but we have $Q(0) = 1$ and $Q(x) = 0$ for $x \neq 0$.

4.3 Unbounded Symbols

If the 'coefficients' $\ell(x), Q(x)$ and $\int_{y\neq 0}(1 \wedge y^2)\ N(x, dy)$ are not bounded in x, some technical difficulties arise along with formula (4.1) of the above theorem. We overcome these problems by using a slightly different formula to calculate the symbol. In the bounded case we get the same $p : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ as above.

Definition 4.3 Let X be an $\mathbb{R}^d$-valued (universal) Markov process, which is conservative and normal. Fix a starting point x and define $T = T_k^x$ to be the first exit time from the ball of radius $k \in \mathbb{R}_+$:

$$T := \inf\{t \ge 0 : \|X_t^x - x\| > k\}.$$

For $\xi \in \mathbb{R}^d$ we call

$$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t^T - x)'\xi} - 1}{t} \tag{4.3}$$

the **symbol** of the process, if the limit exists.

Theorem 4.4 *Let X be an Itô process and let $\ell = (\ell^{(j)})_{1\le j\le d}$ and $Q = (q^{jk})_{1\le j,k\le d}$ be finely continuous and locally bounded; N be such that the function*

$$x \longmapsto \int_{y\neq 0} (1 \wedge y^2)\ N(x,dy)$$

is finely continuous and locally bounded. In this case the limit (4.3) *exists and the symbol of X is*

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y)\right)\ N(x,dy).$$

Remark: This means we investigate the stopped process, which is no longer time-homogeneous Markovian, but has the same behavior as the process X near the starting point x for a short period of time depending on ω. Let us remark that $\mathbb{P}^x(T>0)=1$ since the process is càdlàg. One might at first think that pre-stopping the process is necessary in order to get control over the jumps, but as we are integrating with respect to Lebesgue measure at the vital occasions, it is possible to switch from the càdlàg to the càglàd version of the integrands which solves the problem. This will become clear in the proof. Pre-stopping might, on the other hand, destroy the information of the process: think of a standard Poisson process which is pre-stopped as soon as it leaves the ball of radius $1/2$. This becomes a constant zero process.

Proof: The proof is quite similar to the bounded case. At first we use again Itô's formula to obtain

$$\begin{aligned}
\frac{1}{t}\mathbb{E}^x\left(e^{i(X_t^T - x)'\xi} - 1\right) &= \frac{1}{t}\mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^T - x)'\xi}\ dX_s^T\right)\\
&+\frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^T - x)'\xi}\ d[X^T, X^T]_s^c\right)\\
&+\frac{1}{t}\mathbb{E}^x\left(e^{-ix'\xi}\sum_{0<s\le t}\left(e^{i\xi' X_s^T} - e^{i\xi' X_{s-}^T} - i\xi e^{i\xi' X_{s-}^T}\Delta X_s^T\right)\right).
\end{aligned}$$

The left-continuous process X_{t-}^T is bounded, the stopped jumps $(\Delta X)^T$ are the jumps of the stopped process (ΔX^T) and X^T admits the stopped characteristics:

$$\begin{aligned}
B_t^T(\omega) &= \int_0^{t\wedge T(\omega)} \ell(X_s(\omega))\ ds = \int_0^t \ell(X_s(\omega)) 1_{[[0,T]]}(\omega,s)\ ds\\
C_t^T(\omega) &= \int_0^t Q(X_s(\omega)) 1_{[[0,T]]}(\omega,s)\ ds\\
\nu^T(\omega;ds,dy) &:= 1_{[[0,T]]}(\omega,s)\ N(X_s(\omega),dy)\ ds
\end{aligned}$$

One can now set the integrand at the right endpoint of the stochastic support to zero, as we are integrating with respect to Lebesgue measure:

$$\begin{aligned}
B_t^T(\omega) &= \int_0^t \ell(X_s(\omega))1_{[[0,T[[}(\omega,s)\ ds \\
C_t^T(\omega) &= \int_0^t Q(X_s(\omega))1_{[[0,T[[}(\omega,s)\ ds \\
\nu^T(\omega;ds,dy) &= 1_{[[0,T[[}(\omega,s)\ N(X_s(\omega),dy)\ ds.
\end{aligned}$$

In the first two lines the integrand is now bounded, because ℓ and Q are locally bounded and $\left\|X_s^T(\omega)\right\| < k$ on $[0,T(\omega)[$ for every $\omega \in \Omega$. The rest of the proof is very similar to the one in the bounded case. For the martingale preservation in the first term we obtain

$$\begin{aligned}
\left[e^{i(X^T-x)'\xi} \bullet X^{T,c}, e^{i(X^T-x)'\xi} \bullet X^{T,c}\right]_t &= \left[e^{i(X^T-x)'\xi} \bullet X^{c,T}, e^{i(X^T-x)'\xi} \bullet X^{c,T}\right]_t \\
&= \left[(e^{i(X^T-x)'\xi} \bullet X^c)^T, (e^{i(X^T-x)'\xi} \bullet X^c)^T\right]_t \\
&= \left[e^{i(X^T-x)'\xi} \bullet X^c, e^{i(X^T-x)'\xi} \bullet X^c\right]_t^T \\
&= \left(\int_0^t (e^{i(X_s^T-x)'\xi})^2\ d[X^c,X^c]_s\right)^T \\
&= \int_0^t (e^{i(X_s^T-x)'\xi})^2 1_{[[0,T]]}(s)\ d[X^c,X^c]_s \\
&= \int_0^t (e^{i(X_s^T-x)'\xi})^2 1_{[[0,T]]}(s)\ d\left(\int_0^s Q(X_r)\ dr\right) \\
&= \int_0^t \left((e^{i(X_s^T-x)'\xi})^2 1_{[[0,T[[}(s) Q(X_s)\right)\ ds
\end{aligned}$$

where we used several well known facts about the square bracket. The last term is uniformly bounded in ω and therefore

$$\left[e^{i(X^T-x)'\xi} \bullet X^{T,c}, e^{i(X_t^T-x)'\xi} \bullet X_t^{T,c}\right]_t < \infty \qquad \text{for every } t \geq 0.$$

This means that $e^{i(X_t^T-x)'\xi} \bullet X_t^{T,c}$ is an L^2-martingale which is zero at zero and therefore, its expected value is constantly zero. One of the main tools in the proof of the bounded case was this martingale preservation, the other main tool is a path-by-path Lebesgue integral. We illustrate how to deal with this in the given setting by looking at the second term. Here we have

$$\begin{aligned}
[X^T,X^T]_t^c = [X,X]_t^{c,T} = [X^c,X^c]_t^T = C_t^T &= (Q(X_t) \bullet t)^T \\
&= (Q(X_t) \cdot 1_{[[0,T]]}(t)) \bullet t \\
&= (Q(X_t) \cdot 1_{[[0,T[[}(t)) \bullet t.
\end{aligned}$$

and by Lemma 2.51 a)

$$-\lim_{t\downarrow 0}\frac{1}{2}\xi^2\frac{1}{t}\mathbb{E}^x\int_0^t \underbrace{e^{i(X_s-x)'\xi}}_{\to 1}\underbrace{Q(X_s)}_{\to Q(x)}\underbrace{1_{[[0,T[[}(s)}_{\to 1}\ ds = -\frac{1}{2}\xi^2 Q(x).$$

For the drift part in the first term we obtain analogously

$$\lim_{t\downarrow 0} i\xi\frac{1}{t}\mathbb{E}^x\int_0^t \underbrace{e^{i(X_s-x)'\xi}}_{\to 1}\underbrace{\ell(X_s)}_{\to \ell(x)}\underbrace{1_{[[0,T[[}(s)}_{\to 1}\ ds = i\xi\ell(x).$$

What remains are again the jump parts. We have

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y}-1-i\xi' y\chi(y))\right)1_{[[0,T[[}(\omega,s)\ \mu^X(\cdot;ds,dy)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y}-1-i\xi' y\chi(y))\right)1_{[[0,T[[}(\omega,s)\ \nu(\cdot;ds,dy)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)'\xi}(e^{i\xi' y}-1-i\xi' y\chi(y))\right)1_{[[0,T[[}(\omega,s)\ N(X_s(\omega),dy)\ ds.
\end{aligned}$$

Since X is bounded on $[[0,T[[$ and $x\longmapsto\int_{y\neq 0}(1\wedge y^2)\ N(x,dy)$ is locally bounded, we obtain by (3.7) and Lemma 2.51 a)

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\int_0^t e^{i(X_{s-}-x)'\xi}\left(\int_{y\neq 0}\left(e^{i\xi' y}-1-i\xi' y\cdot\chi(y)\right)1_{[[0,T[[}(\omega,s)\ N(X_s(\omega),dy)\right)\ ds\\
&\quad\xrightarrow[t\downarrow 0]{}\int_{y\neq 0}\left(e^{iy'\xi}-1-iy'\xi\cdot\chi(y)\right)\ N(x,dy)
\end{aligned}$$

which completes the proof. □

Remark: The proof shows in particular, that the limit does not depend on the choice of the radius k. In fact we could have chosen an arbitrary compact set containing a neighborhood of x instead of $\overline{B_k(x)}$.

Corollary 4.5 *Let X be a nice Feller process with (analytic) symbol*

$$p(x,\xi) = -i\ell(x)'\xi+\frac{1}{2}\xi' Q(x)\xi-\int_{y\neq 0}\left(e^{iy'\xi}-1-iy'\xi\cdot\chi(y)\right)\ N(x,dy).$$

and let $\ell=(\ell^{(j)})_{1\le j\le d}$, $Q=(q^{jk})_{1\le j,k\le d}$ and $\int_{y\neq 0}(1\wedge y^2)\ N(\cdot,dy)$ be finely continuous. In this case the limit (4.3) exists and is equal to $p(x,\xi)$.

Proof: By Theorem 3.10 every nice Feller process is an Itô process and the differential characteristics are equal to the Lévy triplet of the symbol. The (analytic) symbol of a nice Feller process is always locally bounded (see Corollary 1.21 and Definition 1.11) and therefore, Lemmas 3.2 - 3.4 imply the assumption of Theorem 4.4. □

4.4 Properties of the Symbol

The analytic symbol which has been introduced in Definition 1.11 is always a (continuous) negative definite function. The following theorem shows, that the (probabilistic) symbol is contained in this class, too.

Theorem 4.6 *Consider the limit* (4.3).

a) If this limit exists pointwise in ξ, we obtain that $\xi \longmapsto p(x,\xi)$ is a negative definite function for every $x \in \mathbb{R}^d$.

b) If this limit exists locally uniform in ξ, we even have $p(x,\cdot) \in CN(\mathbb{R}^d)$, i.e. it is a continuous negative definite function.

The same is true for the limit (4.1).

Remark: In the case of Itô processes the limit has a Lévy-Khinchine representation. This shows that it is indeed a continuous negative definite function.

Proof: Fix $x \in \mathbb{R}^d$. For every $t > 0$ the function $\xi \longmapsto \mathbb{E}^x e^{i(X_t^T - x)'\xi}$ is the characteristic function of the random variable $X_t^T - x$. Therefore, it is a continuous positive definite function. By Proposition 1.2 h) we conclude that $\xi \longmapsto -(\mathbb{E}^x e^{i(X_t^T-x)'\xi} - 1)$ is a continuous negative definite function. Dividing by t does no harm since $N(\mathbb{R}^d)$ and $CN(\mathbb{R}^d)$ are convex cones. The result follows from Proposition 1.2 a) resp. e) □

Next we deal with independent components. Compare in this context the remark after Proposition 1.26.

Lemma 4.7 *Let X be a d-dimensional vector of independent Itô processes $X^{(j)}$ with symbols $p^{(j)}$, $j = 1, ..., d$. The process X admits the symbol*

$$p(x,\xi) = p^{(1)}(x^{(1)}, \xi^{(1)}) + ... + p^{(d)}(x^{(d)}, \xi^{(d)}).$$

Proof: We give the proof for two components. The general case follows inductively. Let X and Y be independent Itô processes with symbols $p(x,\xi_1)$ resp. $q(y,\xi_2)$, where the sum of the dimensions of x and y is d, and consider:

$$\begin{aligned}
&\mathbb{E}^{(x,y)} \frac{e^{i(X_t-x)'\xi_1 + i(Y_t-y)'\xi_2} - 1}{t} \\
&= \frac{\mathbb{E}^{(x,y)}\left(e^{i(X_t-x)'\xi_1 + i(Y_t-y)'\xi_2}\right) - 1}{t} \\
&= \frac{\mathbb{E}^{x}\left(e^{i(X_t-x)'\xi_1}\right) \cdot \mathbb{E}^{y}\left(e^{i(Y_t-y)'\xi_2}\right) - 1}{t} \\
&= \frac{\mathbb{E}^{x}\left(e^{i(X_t-x)'\xi_1}\right) \cdot \mathbb{E}^{y}\left(e^{i(Y_t-y)'\xi_2}\right) - \mathbb{E}^{y}\left(e^{i(Y_t-y)'\xi_2}\right) + \mathbb{E}^{y}\left(e^{i(Y_t-y)'\xi_2}\right) - 1}{t} \\
&= \underbrace{\frac{\mathbb{E}^{x}\left(e^{i(X_t-x)'\xi_1}\right) - 1}{t}}_{\to -p(x,\xi_1)} \cdot \underbrace{\mathbb{E}^{y}\left(e^{i(Y_t-y)'\xi_2}\right)}_{\to 1} + \underbrace{\frac{\mathbb{E}^{y}\left(e^{i(Y_t-y)'\xi_2}\right) - 1}{t}}_{\to -q(y,\xi_2)}.
\end{aligned}$$

Hence the result. □

5 The Symbol Associated to the Solution of an SDE

In this chapter we calculate the symbols associated to the solutions of different kinds of stochastic differential equations.

5.1 Lévy-driven SDE. The Symbol of the Solution

Let Z be an n-dimensional Lévy process (starting at zero) with symbol ψ and consider the following SDE:

$$\begin{aligned} dX_t &= \Phi(X_{t-})\ dZ_t \\ X_0 &= x \end{aligned} \tag{5.1}$$

where $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ is globally Lipschitz continuous. Since Z_t takes its values in $\mathbb{R}^n$ and since the solution X is $\mathbb{R}^d$-valued, formula (5.1) is a short form for the following system of equations

$$X^{x,(j)} = x^{(j)} + \sum_{k=1}^{n} \int \Phi(X_-)^{jk}\ dZ^{(k)}$$

where $j \in \{1,..,d\}$.

Theorem 5.1 *Let $X_t^x(\omega) = X(x,\omega,t)$ be the solution of the SDE* (5.1). *For this process the limit* (4.3) *exists and we have*

$$p(x,\xi) = \psi(\Phi(x)'\xi)$$

where Φ is the coefficient of the SDE and ψ is the symbol of the driving process.

Proof: We have already seen in Section 2.6 that a unique solution of this SDE exists under the Lipschitz condition. Furthermore, this solution is conservative. Here we give the one-dimensional proof for the structure of the symbol. The multidimensional version is proved in the same way. Just the notation becomes more involved. Fix $x, \xi \in \mathbb{R}$. At first we apply Itô's formula to the function $\exp(i(\cdot - x)\xi)$:

$$\begin{aligned} &\frac{1}{t}\mathbb{E}^x\left(e^{i(X_t^T - x)\xi} - 1\right) = \\ &\quad \frac{1}{t}\mathbb{E}^x\Bigg(\int_{0+}^t i\xi\cdot e^{i(X_{s-}^T - x)\xi}\ dX_s^T + \frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^T - x)\xi}\ d[X^T, X^T]_s^c \\ &\quad + e^{-ix\xi}\sum_{0<s\le t}\left(e^{iX_s^T\xi} - e^{iX_{s-}^T\xi} - i\xi e^{iX_{s-}^T\xi}\Delta X_s^T\right)\Bigg). \end{aligned}$$

For the first term we get

$$
\begin{aligned}
&\frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi e^{i(X_{s-}^T - x)\xi}\right) dX_s^T \\
&\quad = \frac{1}{t}\mathbb{E}^x \int_{0+}^t \left(i\xi \cdot e^{i(X_{s-}^T - x)\xi}\right) d\left(\int_0^s \Phi(X_{r-})1_{[[0,T]]}(\cdot, r)\, dZ_r\right) \\
&\quad = \frac{1}{t} \cdot \mathbb{E}^x \int_{0+}^t \Big(i\xi \underbrace{\cdot e^{i(X_{s-}^T - x)\xi}\Phi(X_{s-})1_{[[0,T]]}(\cdot, s)}_{\text{bounded}}\Big)\, dZ_s \\
&\quad = \frac{1}{t} \cdot \mathbb{E}^x \int_{0+}^t \left(i\xi \cdot e^{i(X_{s-}^T - x)\xi}\Phi(X_{s-})1_{[[0,T]]}(\cdot, s)\right) d\left(\ell s + \sum_{0<r\le s} \Delta Z_r 1_{\{|\Delta Z_r| \ge 1\}}\right)
\end{aligned}
$$

where we used the Lévy-Itô-decomposition (Theorem 1.24), in which two of the four terms are martingales. As the integrand is bounded we obtain that the integral with respect to them is again a martingale (see Corollary 2.42) and because these integrals are starting in zero the expected value is constantly zero.

Putting the big jumps aside for the moment we obtain for the drift part

$$
\begin{aligned}
&\frac{1}{t} \cdot \mathbb{E}^x \int_{0+}^t \left(i\xi \cdot e^{i(X_{s-}^T - x)\xi}\Phi(X_{s-})1_{[[0,T]]}(\cdot, s)\ell\right) ds \\
&\qquad = i\xi\ell \cdot \mathbb{E}^x \frac{1}{t}\int_0^t \left(e^{i(X_s^T - x)\xi}\Phi(X_s)1_{[[0,T[[}(\cdot, s)\right) ds \\
&\qquad \xrightarrow[t\downarrow 0]{} i\xi\ell\Phi(x).
\end{aligned}
$$

Since we are integrating with respect to Lebesgue measure the integral becomes a usual (non stochastic) integral, which can be calculated path-by-path. Almost every trajectory of the integrand is càdlàg. This means that the number of jumps is countable and hence the set of jump times is a Lebesgue null set. Therefore, we can switch to the càdlàg version of the integrand, without changing the integral value. To obtain the limit, we have used Lemma 2.51 a).

The calculation of the second term works alike, but we have to calculate the square bracket of the process first:

$$
\begin{aligned}
[X^T, X^T]_t^c = ([X, X]_t^c)^T &= ([(\Phi(X_-) \bullet Z), (\Phi(X_-) \bullet Z)]_t^c)^T \\
&= \int_0^t (\Phi(X_{s-}))^2 1_{[[0,T]]}(\cdot, s)\, d[Z, Z]_s^c \\
&= \int_0^t (\Phi(X_{s-}))^2 1_{[[0,T]]}(\cdot, s)\, d(Qs).
\end{aligned}
$$

Here we made use of Proposition 2.31 g) (cf. the proof of Theorem 4.4). The continuous part of the square bracket was introduced in Section 2.4. In Example 2.38 it was calculated for a Lévy process. Now we can calculate the second term:

$$\begin{aligned}
&\frac{1}{t}\cdot\frac{1}{2}\cdot\mathbb{E}^x\int_{0+}^t\Big(-\xi^2 e^{i(X_{s-}^T-x)\xi}\Big)\,d[X^T,X^T]_s^c\\
&\quad=\frac{1}{t}\cdot\frac{1}{2}\cdot\mathbb{E}^x\int_{0+}^t\Big(-\xi^2 e^{i(X_{s-}^T-x)\xi}\Big)\,d\left(\int_0^s(\Phi(X_{r-}))^2 1_{[[0,T]]}(\cdot,r)Q\,dr\right)\\
&\quad=-\frac{1}{2}\xi^2 Q\mathbb{E}^x\frac{1}{t}\int_0^t\Big(e^{i(X_s^T-x)\xi}\cdot(\Phi(X_s))^2 1_{[[0,T[[}(\cdot,s)\Big)\,ds\\
&\quad\xrightarrow[t\downarrow 0]{}-\frac{1}{2}\xi^2 Q\Phi(x)^2.
\end{aligned}$$

Finally for the third term we use again Proposition 2.31 g):

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x e^{-ix\xi}\sum_{0<s\le t}\Big(e^{iX_{s-}^T\xi}\Big(e^{i\Delta X_s^T\xi}-1-i\xi\Delta X_s^T\Big)\Big)\\
&\quad=\frac{1}{t}\mathbb{E}^x\sum_{0<s\le t}\Big(e^{i(X_{s-}^T-x)\xi}\big(e^{i\Phi(X_{s-})1_{[[0,T]]}(\cdot,s)\Delta Z_s\xi}-1-i\xi\Phi(X_{s-})1_{[[0,T]]}(\cdot,s)\Delta Z_s\big)\Big).
\end{aligned}$$

Comparing this with the remainder term of the first term,

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\int_{0+}^t\Big(i\xi\cdot e^{i(X_{s-}^T-x)\xi}\Phi(X_{s-})1_{[[0,T]]}(\cdot,s)\Big)\,d\left(\sum_{0<r\le s}\Delta Z_r 1_{\{|\Delta Z_r|\ge 1\}}\right)\\
&\quad=\frac{1}{t}\mathbb{E}^x\sum_{0<s\le t}\Big(e^{i(X_{s-}^T-x)\xi}\big(i\xi\Phi(X_{s-})1_{[[0,T]]}(\cdot,s)\Delta Z_s 1_{\{|\Delta Z_s|\ge 1\}}\big)\Big),
\end{aligned}$$

and adding these two terms, we obtain

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\sum_{0<s\le t}\Big(e^{i(X_{s-}^T-x)\xi}\big(e^{i\Phi(X_{s-})1_{[[0,T]]}(\cdot,s)\Delta Z_s\xi}-1-i\xi\Phi(X_{s-})1_{[[0,T]]}(\cdot,s)\Delta Z_s 1_{\{|\Delta X_s|<1\}}\big)\Big)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\mathbb{R}\setminus\{0\}}\big(H_{x,\xi}(\omega,s-,y)\big)\,\mu^X(\cdot;ds,dy)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\mathbb{R}\setminus\{0\}}\big(H_{x,\xi}(\omega,s-,y)\big)\,\nu(\cdot;ds,dy)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_0^t\int_{\mathbb{R}\setminus\{0\}}\big(H_{x,\xi}(\omega,s,y)\big)\,N(dy)\,ds\\
&\quad\xrightarrow[(t\downarrow 0)]{}\int_{\mathbb{R}\setminus\{0\}}\big(e^{i\Phi(x)y\xi}-1-i\xi\Phi(x)y1_{\{|y|<1\}}\big)\,N(dy)
\end{aligned}$$

where we have used the abbreviation

$$H_{x,\xi}(\omega,s,y):=e^{i(X_s^T-x)\xi}\big(e^{i\Phi(X_s)1_{[[0,T]]}(\cdot,s)y\xi}-1-i\xi\Phi(X_s)1_{[[0,T]]}(\cdot,s)y1_{\{|y|<1\}}\big).$$

Here we used Theorem 2.21 a), which allows us to integrate 'under the expectation' with respect to the compensated measure $\nu(\cdot;ds,dy)$ instead of the measure itself. In the case of a Lévy process the compensator has a nice structure as we have seen in Section 2.2. In the end we obtain

$$\begin{aligned} p(x,\xi) &= -i\ell(\Phi(x)\xi) + \frac{1}{2}(\Phi(x)\xi)Q(\Phi(x)\xi) \\ &\quad - \int_{y\neq 0} \left(e^{i(\Phi(x)\xi)y} - 1 - i(\Phi(x)\xi)y \cdot 1_{\{|y|<1\}}(y)\right) N(dy) \\ &= \psi(\Phi(x)\xi). \end{aligned}$$

□

Remarks: a) Note that in the multi-dimensional case the matrix $\Phi(x)$ has to be transposed, i.e. the symbol of the solution is $\psi(\Phi(x)'\xi)$.

b) If the coefficient Φ is bounded, the limit (4.1) exists and is equal to (4.3).

Putting these results together with Theorem 2.49 we obtain the following:

Corollary 5.2 *For every negative definite symbol having the following structure:*

$$p(x,\xi) = \psi(\Phi(x)'\xi)$$

where $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a continuous negative definite function and $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ is bounded and Lipschitz continuous, there exists a unique corresponding Feller process X^x.

In the spirit of Theorem 2.48 this is somehow best possible. As soon as the driving term is not Lévy, the solution is no longer Markovian. However, the bivariate process consisting of the solution and the driving term is a Markovian semimartingale and admits a symbol. We will investigate this in the following section.

5.2 Itô-driven SDE. The Bivariate Symbol

We will now investigate the case where the driving term in equation (5.1) is no longer a Lévy, but a general Itô process, with finely continuous differential characteristics, i.e. for $y \in \mathbb{R}^n$ we consider

$$\begin{aligned} dX_t &= \Phi(X_{t-})\, dY_t^y \\ X_0 &= x \end{aligned}$$

where $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ is Lipschitz continuous or 'acceptable' in the sense of [13]. The solution $X^{x,y}$ depends on y and is not Markovian any more, see Theorem 2.48. However, using Theorem 2.52, we obtain that the $(d+n)$-dimensional process $(X^{x,y}, Y^y)$ is a (universal) strong Markov process and the transition semigroup is given by

$$P_t((x,y),B) = \mathbb{P}^{x,y}((X_t,Y_t) \in B)$$

for $x, y \in \mathbb{R}$, $t \geq 0$ and $B \in \mathcal{B}^{d+n}$. Furthermore, we have the following theorem:

Theorem 5.3 *If $Y = (Y^y)_{y\in\mathbb{R}}$ is an Itô process and $X^{x,y}$ is the solution of the SDE*

$$\begin{aligned} dX_t &= \Phi(X_{t-})\, dY_t^y \\ X_0 &= x \end{aligned} \tag{5.2}$$

then the $(d+n)$-dimensional process $(X,Y) = (X^{x,y}, Y^y)$ is an Itô process.

Proof: We already know that the process (X,Y) is a Markov semimartingale (see Theorem 2.52). Let us denote the differential characteristics of Y with respect to $\chi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $(\ell, Q, N(\cdot, dy))$. To obtain the structure of the characteristics we use [36] Proposition IX.5.3, written in a suitable form. It shows that the characteristics $(\tilde{B}, \tilde{c}, \tilde{\nu})$ of the process (X,Y) with respect to the cut-off function $\tilde{\chi} : \mathbb{R}^{d+n} \longrightarrow \mathbb{R}$ are

$$\begin{aligned}
\tilde{B}_t &= \begin{pmatrix} \int_0^t \Phi(X_s)\ell(Y_s)\, ds \\ \int_0^t \ell(Y_s)\, ds \end{pmatrix} \\
&\quad + \int_0^t \int_{\mathbb{R}^n\setminus\{0\}} \begin{pmatrix} \Phi(X_{s-})z \\ z \end{pmatrix} \left(\tilde{\chi}\begin{pmatrix} \Phi(X_{s-})z \\ z \end{pmatrix} - \chi(z) \right) N(Y_s, dz)\, ds \\
\tilde{C}_t &= \left(\begin{array}{c|c} \int_0^t \Phi(X_s)Q(Y_s)(\Phi(X_s))'\, ds & \int_0^t \Phi(X_s)Q(Y_s)\, ds \\ \hline \int_0^t Q(Y_s)(\Phi(X_s))'\, ds & \int_0^t Q(Y_s)\, ds \end{array} \right) \in \left(\begin{array}{c|c} \mathbb{R}^{d\times d} & \mathbb{R}^{d\times n} \\ \hline \mathbb{R}^{n\times d} & \mathbb{R}^{n\times n} \end{array} \right) \\
\tilde{\nu}(\omega, ds, dz) &= f_*^{\omega,s}(N(Y_s(\omega), dz))\, ds
\end{aligned}$$

where

$$f_*^{\omega,s}(N(Y_s(\omega), dz)) = N(Y_s(\omega), f^{\omega,s} \in dz)$$

and

$$f_*^{\omega,s}(y) := \begin{pmatrix} \Phi(X_{s-}(\omega))y \\ y \end{pmatrix}.$$

We switched to the càdlàg version of X using the same argument as in the proof of Theorem 5.1. Therefore, the $(d+n)$-dimensional process is again Itô. □

Remark: A possible way to choose the cut-off functions in different dimensions $m \in \mathbb{N}$ is as follows: take a one-dimensional cut-off function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ and define for $x \in \mathbb{R}^m$: $\tilde{\chi}(x) := \chi(x^{(1)}) \cdots \chi(x^{(m)})$.

As the process (X,Y) is an Itô process, it admits a symbol.

Theorem 5.4 *If in the above setting Φ is Lipschitz continuous or if it is acceptable and finely continuous, then the limit* (4.3) *exists for the process (X,Y). The symbol $q : \mathbb{R}^{d+n} \times \mathbb{R}^{d+n} \longrightarrow \mathbb{C}$ is*

$$q\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right) = p\left(y, \left(\begin{array}{c|c} (\Phi(x))' & I_n \end{array} \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right)$$

where I_n is the $n \times n$-identity matrix, Φ is the coefficient of the SDE and $p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ is the symbol of the driving Itô process.

Proof: Either one uses the characteristics of the bivariate process given above and Theorem 4.4 or one uses a calculation which is similar to the one in the previous section. □

Like Theorem 4.1 this result shows that in the context of SDEs the symbol is at least as natural an object to study as the semimartingale characteristics.

5.3 The Symbol of a General SDE

First we give another characterization of Itô processes which was established in [12]. To this end we need the following definition:

Definition 5.5 Let

$$\mathbf{X} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, (\vartheta_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$$

be a strong Markov process and $(\Omega', \mathcal{F}', \mathbb{P}')$ be an auxiliary probability space. Set

$$(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}^x) := (\Omega, \mathcal{F}, \mathbb{P}^x) \otimes (\Omega', \mathcal{F}', \mathbb{P}')$$

and let π be the projection mapping $(\omega, \omega') \longrightarrow \omega$ from $\widetilde{\Omega}$ to Ω. For any random variable Z defined on Ω (e.g. $Z = X_t$), we denote by the same letter Z its natural extension $Z \circ \pi$ to $\widetilde{\Omega}$: $Z(\omega, \omega') = Z(\omega)$ and similarly for Y defined on Ω'.
Let $(\widetilde{\mathcal{F}}_t)_{t\geq 0} = \widetilde{\mathbb{F}}$ be a filtration on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ which is right continuous and $(\widetilde{\vartheta}_t)_{t\geq 0}$ be a semigroup of transformations on $\widetilde{\Omega}$.
The process

$$\widetilde{\mathbf{X}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{\vartheta}_t)_{t\geq 0}, (X_t)_{t\geq 0}, \widetilde{\mathbb{P}}^x)_{x\in\mathbb{R}^d}$$

is called a **strong Markov extension** of $\mathbf{X}$ if

(i) π is $\widetilde{\mathcal{F}}_t/\mathcal{F}_t$-measurable and $\pi \circ \widetilde{\vartheta}_t = \vartheta_t \circ \pi$ for all $t \geq 0$.

(ii) For every $Z \in B_b(\widetilde{\mathcal{F}})$ and every finite stopping time T of $\widetilde{\mathbb{F}}$, $Z \circ \widetilde{\vartheta}_T$ is measurable with respect to the completion of $\widetilde{\mathcal{F}}$ and

$$\widetilde{\mathbb{E}}^x(Z \circ \widetilde{\vartheta}_T | \widetilde{\mathcal{F}}_T) = \widetilde{\mathbb{E}}^{X_T}(Z)$$

(conditional expectation with respect to $\widetilde{\mathbb{P}}$).

Remark: The process $\widetilde{\mathbf{X}}$ is then strong Markov and has the same transition function as $\mathbf{X}$; only the probability space (in the background) changes:

$$\widetilde{\mathbb{P}}(X_t(\omega, \omega') \in B | X_s(\omega, \omega') = x) = P_{s,t}(x, B) = \mathbb{P}(X_t(\omega) \in B | X_s(\omega) = x).$$

Proposition 5.6 *X is an Itô process if and only if there exists a strong Markov extension $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ supporting a d-dimensional Brownian motion $\widetilde{W}$ and a Poisson random measure $\widetilde{\mu}$ on $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$ with compensator $dtN(dz)$ such that X is the solution of the SDE*

$$\begin{aligned} X_t &= x + \int_0^t \ell(X_s)\ ds + \int_0^t \sigma(X_s)\ d\widetilde{W}_s \\ &+ \int_0^t \int_{\mathbb{R}} k(X_{s-}, z) 1_{\{|k(X_{s-},z)| \leq 1\}} \left(\widetilde{\mu}(ds, dz) - ds\ N(dz) \right) \\ &+ \int_0^t \int_{\mathbb{R}} k(X_{s-}, z) 1_{\{|k(X_{s-},z)| > 1\}}\ \widetilde{\mu}(ds, dz) \end{aligned} \tag{5.3}$$

$\widetilde{\mathbb{P}}^x$-a.s. for every $x \in \mathbb{R}^d$ for some $(\mathcal{B}^d)^*$-measurable functions $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times d}$ and

$$k : (\mathbb{R}^d \times \mathbb{R}, (\mathcal{B}^d)^* \otimes \mathcal{B}^1) \longrightarrow (\mathbb{R}^d \backslash \{0\}, \mathcal{B}(\mathbb{R}^d \backslash \{0\})).$$

In this case the Itô process has the characteristics

$$\begin{aligned} B_t &= \int_0^t \ell(X_s)\ ds \\ C_t &= \int_0^t \sigma(X_s)(\sigma(X_s))'\ ds \\ \nu(\cdot; ds, dy) &= N\Big(k(X_s, \cdot) \in dy\Big)\ ds \end{aligned}$$

with respect to the cut-off function $\chi(y) := y \cdot 1_{\overline{B_1(0)}}(y)$.

Proof: See [12] Theorem (3.33) and Proposition (3.15). The remarks on the conventions of [13] are still in place. The proof relies mainly on the results of N. El Karoui and J.-P. Lepeltier [16] on the conversion of multivariate point processes to Poisson random measures. □

Using our general Theorem 4.4 we obtain in the case of finely continuous differential characteristics:

Theorem 5.7 *Let X be a solution of the SDE* (5.3). *If the differential characteristics are finely continuous and locally bounded the limit* (4.3) *exists and the symbol of the process X is*

$$-i(\ell(x))'\xi + \frac{1}{2}\xi'\sigma(x)(\sigma(x))'\xi + \int_{y \neq 0} \Big(1 - e^{i(k(x,y))'\xi} + i(k(x,y))'\xi \cdot 1_{\overline{B_1(0)}}(k(x,y))\Big)\ N(dy).$$

Remark: The theorem does not say whether or not the SDE (5.3) admits a (unique) solution, but it links this question of existence to the question for which symbols there exist a process and to J. Jacod's semimartingale problem (see [37] (12.9)).

6 Some Applications

In the case of Itô processes we can use the symbol to calculate both: the generator and the characteristics of the process.

Furthermore, if a process admits a symbol, it is possible to introduce so called indices. In the case of nice Feller processes these indices allow us to say something about the limiting behavior of the maximal process and about the strong p-variation.

6.1 Symbol, Generator and Characteristics

In the sequel X is an $\mathbb{R}^d$-valued Itô process with finely continuous differential characteristics as in Theorem 4.4.

Let us emphasize that we have not only established a useful way to calculate the symbol of an Itô process. But we are also able to calculate the (extended) generator for a wide range of processes directly, without knowing what the semigroup looks like. We will use this procedure in Section 7.1 to calculate the generator of the COGARCH process.

We use the second part of Theorem 3.13: the symbol, in its standard representation

$$p(x,\xi) = -i\ell(x)\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \Big(e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y) \Big) \, N(x,dy) \tag{6.1}$$

gives us the triplet $(\ell(x), Q(x), N(x,dy))$. Then we are able to write down the (extended) generator explicitly:

$$\begin{aligned} Au(x) &= \ell(x)'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^{d} q^{jk}(x)\partial_j\partial_k u(x) \\ &\quad + \int_{y\neq 0} \Big(u(x+y) - u(x) - y'\nabla u(x) \cdot \chi(y) \Big) \, N(x,dy) \end{aligned} \tag{6.2}$$

for $u \in D(A_{ext}) \supset C_b^2(\mathbb{R}^d)$. And for the semimartingale characteristics we have

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega)) \, ds \\ C_t^{jk}(\omega) &= \int_0^t q^{jk}(X_s(\omega)) \, ds \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy) \, ds \end{aligned} \tag{6.3}$$

with respect to χ.

The program is as follows:

1.) Use formula (4.3) to calculate the symbol of an Itô process.

2.) Write the symbol in standard representation (6.1).

3.) Determine the triplet $(\ell(x), Q(x), N(x,dy))$.

4.) Use the triplet and formula (6.2) to obtain the extended generator of the process on $C_b^2(\mathbb{R}^d)$.

One might think that the critical point in these considerations is 2.) because it could be hard to get the symbol in standard representation. However, in virtually all examples the standard representation appears almost at once.

Remark: Symbol and Characteristics
Suppose that an Itô process is given as in the Theorem 3.13 (iii). In order to find the version of the characteristics, where the process itself appears in the characteristics (see Definition 2.25), we can calculate again the symbol, write it in the standard representation (6.1) and plug the differential characteristics into (6.3).

6.2 Indices

In [56] R. L. Schilling introduced so called indices of symbols. Theses are generalizations of the indices which were introduced by R. M. Blumenthal and R. K. Getoor in [8] and W. E. Pruitt in [51]. These indices can be used to obtain properties of the paths of the corresponding process. Most of the results concerning this relationship are (by now) restricted to the class of Feller processes. However, we strongly believe that most of them can be generalized to Itô processes. The following remark and definition are taken from [56]:

Remark: It is shown in [31] Lemma 5.2 that

$$\frac{\|y\|^2}{1+\|y\|^2} = \int_{\rho\neq 0} \Big(1-\cos(y'\rho)\Big) g(\rho)\, d\rho$$

where (for $\rho \in \mathbb{R}^d \backslash \{0\}$)

$$g(\rho) = \frac{1}{2}\int_0^\infty (2\pi y)^{-d/2} e^{-\|\rho\|^2/(2y)} e^{-2/y}\, dy. \tag{6.4}$$

Definition 6.1 Let $p(x,\xi)$ be a continuous negative definite function. We define (for $x \in \mathbb{R}^d$ and $R > 0$)

$$H(x,R) := \sup_{\|y-x\|\leq 2R} \sup_{\|\varepsilon\|\leq 1} \left(\int_{-\infty}^{\infty} \operatorname{Re} p\left(y, \frac{\rho\varepsilon}{R}\right) g(\rho)\, d\rho + \left| p\left(y, \frac{\varepsilon}{R}\right)\right| \right)$$

with the function g of (6.4). Furthermore, if $|\operatorname{Im} p(x,\xi)| \leq c_0 \cdot \operatorname{Re} p(x,\xi)$, let

$$h(x,R) := \inf_{\|y-x\|\leq 2R} \sup_{\|\varepsilon\|\leq 1} \operatorname{Re} p\left(y, \frac{\varepsilon}{4\kappa R}\right)$$

where $\kappa := (4\arctan(1/2c_0))^{-1}$ and we set

$$h(R) := \inf_{x\in\mathbb{R}^d} h(x,R) \qquad \text{and} \qquad H(R) := \sup_{x\in\mathbb{R}^d} H(x,R).$$

Then the quantities

$$
\begin{aligned}
\beta_0 \quad &:= \sup\{\lambda \geq 0 : \limsup_{R\to\infty} R^\lambda H(R) = 0\}\\
\underline{\beta}_0 \quad &:= \sup\{\lambda \geq 0 : \liminf_{R\to\infty} R^\lambda H(R) = 0\}\\
\overline{\delta}_0 \quad &:= \sup\{\lambda \geq 0 : \limsup_{R\to\infty} R^\lambda h(R) = 0\}\\
\delta_0 \quad &:= \sup\{\lambda \geq 0 : \liminf_{R\to\infty} R^\lambda h(R) = 0\}
\end{aligned}
$$

are called **indices of $p(x,\xi)$ at the origin**. And for each fixed $x \in \mathbb{R}^d$

$$
\begin{aligned}
\beta^x_\infty \quad &:= \inf\{\lambda > 0 : \limsup_{R\to 0} R^\lambda H(x,R) = 0\}\\
\underline{\beta}^x_\infty \quad &:= \inf\{\lambda > 0 : \liminf_{R\to 0} R^\lambda H(x,R) = 0\}\\
\overline{\delta}^x_\infty \quad &:= \inf\{\lambda > 0 : \limsup_{R\to 0} R^\lambda h(x,R) = 0\}\\
\delta^x_\infty \quad &:= \inf\{\lambda > 0 : \liminf_{R\to 0} R^\lambda h(x,R) = 0\}
\end{aligned}
$$

are the **indices of $p(x,\xi)$ at infinity**.

Note that for symmetric α-stable processes (see e.g. [53]) all of the above indices are equal to α. We use the following proposition in order to prove a characterization of the index β^x_∞.

Proposition 6.2 *The only continuous negative definite function vanishing at infinity is constantly zero.*

Proof: Let ψ be a continuous negative definite function which vanishes at infinity: let $\varepsilon > 0$. There exists a radius $R > 0$ such that $\psi(\xi) \leq \varepsilon/4$ if $\xi \in B_R(0)^c$. For every $\gamma \in B_R(0)$ there exist two vectors $\xi, \eta \in B_R(0)^c$ such that $\gamma = \xi + \eta$. By the sub-additivity of $\sqrt{|\psi|}$ we obtain

$$
\sqrt{|\psi(\gamma)|} = \sqrt{|\psi(\xi+\eta)|} \leq \sqrt{|\psi(\xi)|} + \sqrt{|\psi(\eta)|} \leq 2 \cdot \sqrt{\varepsilon/4}
$$

which completes the proof. □

Theorem 6.3 *For every $x \in \mathbb{R}^d$ such that $p(x,\xi)$ is not constantly zero, the index β^x_∞ can be written in the following way*

$$
\beta^x_\infty = \tilde{\beta}^x := \limsup_{\|\eta\|\to\infty} \sup_{\|y-x\|\leq 2/\|\eta\|} \frac{\log|p(y,\eta)|}{\log\|\eta\|}.
$$

Remark: This is a concept which appears in the theory of regular variation (cf. [7] pages 73-74).

Proof: At first we show that the right-hand side is in $[0,2]$: fix $x \in \mathbb{R}^d$. For $\|\eta\| > 1$ we have only to consider points y such that $\|y - x\| \leq 2$. Since the symbol is locally bounded we obtain (cf. Corollary 1.21 and Lemma 3.2)

$$
\frac{\log|p(y,\eta)|}{\log\|\eta\|} \leq \frac{\log(2C_{\overline{B_2(x)}}) + \log\|\eta\|^2}{\log\|\eta\|} \leq \frac{\log(2C_{\overline{B_2(x)}})}{\log\|\eta\|} + 2
$$

which tends to 2 as $\|\eta\|$ goes to infinity. On the other hand we have

$$\sup_{\|y-x\|\leq 2/\|\eta\|} \frac{\log|p(y,\eta)|}{\log\|\eta\|} \geq \frac{\log|p(x,\eta)|}{\log\|\eta\|}.$$

On the first sight, this last term could tend to $-\infty$. We show in the following that this is not the case: using Proposition 6.2 we obtain that there exists a $\delta > 0$ such that for every $R > 0$ there exists a ξ with $\|\xi\| \geq R$ and $|p(x,\xi)| > \delta$. Therefore, we have

$$\limsup_{\|\eta\|\to\infty} \frac{\log|p(x,\eta)|}{\log\|\eta\|} \geq \limsup_{\|\eta\|\to\infty} \frac{\log\delta}{\log\|\eta\|}$$

since the numerator is bounded from below this lim sup is zero.
It remains to show that

$$\limsup_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\leq 2/\|\xi\|}|p(x,\xi)|}{\|\xi\|^{\lambda}}$$

is zero for every $\lambda > \tilde{\beta}^x$ and infinity for every $\lambda < \tilde{\beta}^x$. Let $\varepsilon \in \mathbb{R}$:

$$\begin{aligned}
&\frac{\sup_{\|x-y\|\leq 2/\|\xi\|}|p(x,\xi)|}{\|\xi\|^{\tilde{\beta}^x+\varepsilon}} \\
&= \exp\left(\log\left(\sup_{\|x-y\|\leq 2/\|\xi\|}|p(x,\xi)|\right) - \left(\tilde{\beta}^x+\varepsilon\right)\log\|\xi\|\right) \\
&= \exp\left(\left(\frac{\sup_{\|y-x\|\leq 2/\|\xi\|}\log|p(y,\xi)|}{\log\|\xi\|} - \tilde{\beta}^x\right)\cdot\log\|\xi\| - \varepsilon\cdot\log\|\xi\|\right).
\end{aligned}$$

Taking the lim sup for $\|\xi\|\to\infty$ of this expression, the inner bracket converges to zero since $\tilde{\beta}^x \in [0,2]$ by our above considerations. This means there exists a radius R such that for every $R' \geq R$:

$$\left(\sup_{\|\xi\|\geq R'} \frac{\sup_{\|y-x\|\leq 2/\|\xi\|}\log|p(y,\xi)|}{\log\|\xi\|} - \tilde{\beta}^x\right) < \frac{\varepsilon}{2}.$$

In the end we obtain in the case $\varepsilon > 0$

$$\limsup_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\leq 2/\|\xi\|}|p(x,\xi)|}{\|\xi\|^{\tilde{\beta}^x+\varepsilon}} \leq \limsup_{\|\xi\|\to\infty} \exp(\log(\|\xi\|^{-\varepsilon/2})) = 0$$

and in the case $\varepsilon < 0$

$$\limsup_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\leq 2/\|\xi\|}|p(x,\xi)|}{\|\xi\|^{\tilde{\beta}^x+\varepsilon}} \geq \limsup_{\|\xi\|\to\infty} \exp(\log(\|\xi\|^{-\varepsilon/2})) = \infty$$

which completes the proof. □

As an example we consider in the following the symbol of the Lévy driven SDE which we have calculated in Section 5.1.

Theorem 6.4 *Let X be the solution of the SDE*

$$X_t = x + \int_0^t \Phi(X_{s-})\, dZ_s$$

where Φ is bounded and Lipschitz continuous and the linear mapping $\xi \longmapsto \Phi(y)'\xi$ is bijective for every $y \in \mathbb{R}^d$. Let β_∞^ψ be the index of the driving Lévy process (only depending on ψ and not on x since Z is Lévy). In this case we have for the process X that $\beta_\infty^x = \beta_\infty^\psi$ for every $x \in \mathbb{R}^d$.

Proof: Since the growth condition (3.4) is fulfilled we have the following characterization of β_∞^x (see [56]):

$$\begin{aligned} \beta_\infty^x &= \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\le 2/\|\xi\|} |p(y,\xi)|}{\|\xi\|^\lambda} = 0\right\} \\ &= \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\le 2/\|\xi\|} \sqrt{|p(y,\xi)|}}{\|\xi\|^{\lambda/2}} = 0\right\}. \end{aligned} \tag{6.5}$$

The symbol is in our context: $p(x,\xi) = \psi(\Phi(x)'\xi)$. We write (for $\|\xi\| \ge 1$):

$$\frac{\sup_{\|x-y\|\le 2/\|\xi\|} \sqrt{|\psi(\Phi(y)'\xi)|}}{\|\xi\|^{\lambda/2}} = \frac{\sup_{\|x-y\|\le 2/\|\xi\|} \sqrt{|\psi(\Phi(y)'\xi)|}}{\|\Phi(x)'\xi\|^{\lambda/2}} \cdot \frac{\|\Phi(x)'\xi\|^{\lambda/2}}{\|\xi\|^{\lambda/2}}$$

where the latter factor is bounded above and below, since the linear mapping $\xi \longmapsto \Phi(x)'\xi$ is bijective and bi-continuous. Therefore, we obtain

$$\beta_\infty^x = \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\le 2/\|\xi\|} \sqrt{|\psi(\Phi(y)'\xi)|}}{\|\Phi(x)'\xi\|^{\lambda/2}}\right\}.$$

Denoting the Lipschitz constant of $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ by L we have (if the respective limits exist)

$$\begin{aligned} \lim_{\|\xi\|\to\infty} \frac{\sqrt{|\psi(\Phi(x)'\xi)|}}{\|\Phi(x)'\xi\|^{\lambda/2}} &\le \lim_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\le 2/\|\xi\|} \sqrt{|\psi(\Phi(y)'\xi)|}}{\|\Phi(x)'\xi\|^{\lambda/2}} \\ &\le \lim_{\|\eta\|\to\infty} \frac{\sup_{\varepsilon\le 2L} \sqrt{|\psi(\eta+|\varepsilon|)|}}{\|\eta\|^{\lambda/2}}. \end{aligned} \tag{6.6}$$

The first inequality is trivial whilst the second one is obtained by using the Lipschitz continuity of Φ. Rewriting the first term in (6.6) we obtain

$$\lim_{\|\xi\|\to\infty} \frac{\sqrt{|\psi(\Phi(x)'\xi)|}}{\|\Phi(x)'\xi\|^{\lambda/2}} = \lim_{\|\eta\|\to\infty} \frac{\sqrt{|\psi(\eta)|}}{\|\eta\|^{\lambda/2}}$$

and for the third one we get, by using the sub-additivity of $\sqrt{|\psi(\cdot)|}$,

$$\frac{\sup_{\varepsilon\le 2L} \sqrt{|\psi(\eta+\varepsilon)|}}{\|\eta\|^{\lambda/2}} \le \frac{\sqrt{|\psi(\eta)|}}{\|\eta\|^{\lambda/2}} + \frac{\sup_{\varepsilon\le 2L} \sqrt{|\psi(\varepsilon)|}}{\|\eta\|^{\lambda/2}}$$

where the second term on the right-hand side tends to zero as $\|\eta\|$ goes to infinity. Therefore, we obtain $\beta_\infty^x = \beta_\infty^\psi$. □

We can use this theorem to derive an interesting result on the p-variation of the solution.

Definition 6.5 If $p \in]0, \infty[$ and g is an $\mathbb{R}^d$-valued function on the interval $[a, b]$ then

$$V^p(g; [a, b]) := \sup_{\pi_n} \sum_{j=1}^{n} \|g(t_j) - g(t_{j-1})\|^p$$

where the supremum is taken over all partitions
$\pi_n = (a = t_0 < t_1 < ... < t_n = b)$ is called the **(strong)** p**-variation** of g over $[a, b]$.

Corollary 6.6 *Let $X = (X_t)_{t \geq 0}$ be the solution of the SDE (as in Theorem 6.4)*

$$X_t = x + \int_0^t \Phi(X_{s-})\, dZ_s$$

where Z is a Lévy process with characteristic exponent ψ and Index $\beta_\infty^x = \beta_\infty^\psi$. Then for every $p > \beta_\infty^\psi$ the p-variation of the process X is on every compact time-interval $[0, T]$ a.s. finite.

Proof: The process X is strong Markov. Therefore, we can use a result which is due to M. Manstavičius. Consider for $h \in [0, T]$ and $a > 0$:

$$\begin{aligned}
\alpha(h, a) &= \sup\{\mathbb{P}^x(\|X_t - x\| \geq a) : x \in \mathbb{R}^d, 0 \leq t \leq (h \wedge T)\} \\
&\leq \sup_{t \leq h} \sup_{x \in \mathbb{R}^d} \mathbb{P}^x \left(\sup_{0 \leq s \leq t} \|X_s - x\| \geq a \right).
\end{aligned}$$

Using [56] Lemma 4.1 and Lemma 5.1 we obtain

$$\mathbb{P}^x \left(\sup_{0 \leq s \leq t} \|X_s - x\| \geq a \right) \leq C \cdot t \sup_{\|y - x\| \leq 2a} \sup_{\|\varepsilon\| \leq 1} \left| p\left(y, \frac{\varepsilon}{a}\right) \right|$$

where $C \geq 0$ is independent of x and t. Using this inequality we get

$$\begin{aligned}
\alpha(h, a) &\leq \sup_{t \leq h} \sup_{x \in \mathbb{R}^d} C \cdot t \sup_{\|y - x\| \leq 2a} \sup_{\|\varepsilon\| \leq 1} \left| p\left(y, \frac{\varepsilon}{a}\right) \right| \\
&\leq C \cdot h \sup_{x \in \mathbb{R}^d} \left(\sup_{\|y - x\| \leq 2a} \sup_{\|\varepsilon\| \leq 1} \left| p\left(y, \frac{\varepsilon}{a}\right) \right| \right) \\
&= C \cdot h \sup_{x \in \mathbb{R}^d} \left(\sup_{\|\varepsilon\| \leq 1} \sup_{\|y - x\| \leq 2a} \left| p\left(y, \frac{\varepsilon}{a}\right) \right| \right) \\
&= C \cdot h \sup_{x \in \mathbb{R}^d} \left(\sup_{\|\eta\| \leq (1/a)} \sup_{\|y - x\| \leq (2/\|\eta\|)} |p(y, \eta)| \right).
\end{aligned}$$

By [56] Proposition 5.2 we have that for every $\lambda > \beta_\infty^x = \beta_\infty^\psi$

$$\frac{\sup_{\|y-x\|\leq(2/\|\eta\|)} |p(y,\eta)|}{\|\eta\|^\lambda}$$

converges to zero for η tending to infinity ($a \to 0$) and is hence bounded at infinity. Therefore, for every x there exists a compact set K such that

$$\sup_{\|y-x\|\leq(2/\|\eta\|)} |p(y,\eta)| \leq \tilde{C} \cdot \|\eta\|^\lambda \leq \tilde{C} \cdot a^{-\lambda}$$

on the complement of K. Since the right-hand side is independent of x we finally have

$$\alpha(h,a) \leq C \cdot \tilde{C} \cdot \frac{h^1}{a^\lambda}$$

and obtain that X is in the class $\mathcal{M}(1,\beta_\infty^\psi)$ of M. Manstavičius and the result follows from [45] Theorem 1.3. □

Finally we derive some results concerning the maximum process $(X_\cdot - x)_t^* := \sup_{0\leq s\leq t} \|X_s - x\|$.

Corollary 6.7 *In the setting of Theorem 6.4 we have $\mathbb{P}^x$-a.s. for every $x \in \mathbb{R}^d$:*

$$\begin{aligned} \lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* &= 0 \textit{ for every } \lambda > \beta_\infty^\psi \\ \lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* &= \infty \textit{ for every } \lambda < \beta_\infty^\psi \end{aligned}$$

Proof: This statement follows directly from Theorem 6.4 above and Theorem 4.6 of [56]. □

6.3 Moment Estimates for Feller processes

Theorem 6.8 *Let X be a Feller process with generator A such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$ and symbol $p(x,\xi)$ which fulfills the growth condition* (3.4)*. For $\lambda > 0$ we have that $\mathbb{E}\left(\left((X_t^x - x)^*\right)^\lambda\right)$ is finite if the following integral is finite:*

$$\int_1^\infty r^{\lambda-1} \cdot \sup_{\|y-x\|\leq 2r} \sup_{\|\delta\|\leq 1/r} |p(y,\delta)| \; dr.$$

Proof: Consider for $\lambda > 0$:

$$\begin{aligned} \mathbb{E}\left(\left((X_t^x - x)^*\right)^\lambda\right) &= \lambda \int_0^\infty r^{\lambda-1} \cdot \mathbb{P}((X_t^x - x)^* > r)\; dr \\ &= \lambda \cdot \int_0^1 r^{\lambda-1} \cdot \mathbb{P}((X_t^x - x)^* > r)\; dr + \lambda \cdot \int_1^\infty r^{\lambda-1} \cdot \mathbb{P}((X_t^x - x)^* > r)\; dr \\ &\leq \lambda \cdot \int_0^1 r^{\lambda-1}\; dr + \lambda \cdot \int_1^\infty r^{\lambda-1} \cdot \mathbb{P}^x((X_t - x)^* > r)\; dr \\ &\leq [r^\lambda]_{r=0}^1 + \lambda \cdot c_d \cdot t \cdot \int_1^\infty r^{\lambda-1} \cdot H(x,r)\; dr \\ &\leq 1 + \lambda \cdot c_d \cdot t \cdot 2 \cdot \tilde{c} \int_1^\infty r^{\lambda-1} \sup_{\|y-x\|\leq 2r} \sup_{\|\delta\|\leq 1/r} |p(y,\delta)|\; dr \end{aligned}$$

where the last two inequalities follow from Lemma 4.1 respective Lemma 5.1 of [56] and $H(x, r)$ was introduced in Definition 6.1 above. □

7 Processes Used in Mathematical Finance

In this chapter we are dealing with two processes which are used in applications: the COGARCH process is a process which has been introduced quite recently, while the Ornstein-Uhlenbeck process is a classical example in the theory of stochastic differential equations.

7.1 COGARCH: Symbol and Generator

The COGARCH process was introduced by C. Klüppelberg, A. Lindner and R. Maller in [42]. It is a continuous time analog of the classic GARCH process (in discrete time) based on a single background driving Lévy process. In this section we calculate the symbol of the COGARCH process (and its volatility process) in order to provide a straightforward way to determine the generator.

Recall how the COGARCH process is defined:
We start with a Lévy process $Z = (Z_t)_{t\geq 0}$ with triplet (ℓ, Q, N). Fix $0 < \delta < 1$, $\beta > 0$ and $\lambda \geq 0$. Then the volatility process $(\sigma_t)_{t\geq 0}$ is the solution of the SDE

$$\begin{aligned} d\sigma_t^2 &= \beta \; dt + \sigma_t^2 \left(\log \delta \; dt + \frac{\lambda}{\delta} \; d[Z,Z]_t^{disc} \right) \\ \sigma_0 &= \widetilde{s} \; (> 0) \end{aligned}$$

where

$$[Z,Z]_t^{disc} = \sum_{0<s\leq t} (\Delta Z_s)^2.$$

It turns out that $(\sigma_t)_{t\geq 0}$ is a universal Markov process (on $\mathbb{R}_+ \backslash \{0\}$).

Definition 7.1 The process

$$G_t := g + \int_0^t \sigma_{s-} \; dZ_s, \qquad g \in \mathbb{R}$$

is called **COGARCH process** (starting in g).

Observe that we allow the process to start everywhere in order to apply our methods. The pair $(G, \sigma)'$ is a (normal) Markov process which is homogeneous in time. It is homogeneous in space in the first component. Furthermore, $(G, \sigma^2)'$ is an Itô process, which follows directly from the definition.

To avoid problems which might arise for processes defined on $\mathbb{R} \times (\mathbb{R}_+ \backslash \{0\})$ we consider in the sequel the process : $X := (G, \log(\sigma^2))'$, i.e. $X^{(2)}$ is the logarithmic squared volatility. Using Itô's formula one obtains that X is again an Itô process.

Theorem 7.2 *The process $X = (G_t, \log(\sigma_t^2))'$ admits the symbol $p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{C}$:*

$$
\begin{aligned}
p\,(x,\xi) = & -i\xi^{(1)}\Big(\ell \exp(x^{(2)}/2)\Big)\\
& -i\xi^{(1)}\Big(\exp(x^{(2)}/2)\int_{\mathbb{R}\setminus\{0\}} y\cdot(1_{\left\{\left|\exp(x^{(2)}/2)y\right|<1\right\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}})\ N(dy)\Big)\\
& -i\xi^{(2)}\Big(\tfrac{\beta}{\exp(x^{(2)})}+\log\delta\Big)\\
& -i\xi^{(2)}\Big(\int_{\mathbb{R}\setminus\{0\}}\log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\left\{\left|\exp(x^{(2)}/2)y\right|<1\right\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}})\ N(dy)\Big)\\
& +\tfrac{1}{2}(\xi^{(1)})^2\exp(x^{(2)})Q\\
& -\int_{\mathbb{R}^2\setminus\{0\}}\Big(\exp(i(z_1,z_2)\xi)-1-iz'\xi\cdot(1_{\{|z_1|<1\}}\cdot 1_{\{|z_2|<1\}})\Big)\ f_*^{x^{(2)}}N\,(dz)
\end{aligned}
$$

where $f_*^{x^{(2)}}N$ is the image measure

$$f_*^{x^{(2)}}N\,(B) := N(f^{x^{(2)}}\in B) \qquad \text{for} \qquad B\in\mathcal{B}^2 \tag{7.1}$$

and the function $f^{x^{(2)}} : \mathbb{R}\longrightarrow\mathbb{R}^2$ is given by

$$f^{x^{(2)}} : \eta \longmapsto \begin{pmatrix} e^{x^{(2)}/2}\eta \\ \log(1+(\lambda/\delta)\ \eta^2)\end{pmatrix} \qquad \text{for} \qquad x^{(2)}\in\mathbb{R}. \tag{7.2}$$

The Lévy triplet $(\tilde\ell(x), \tilde Q(x), \tilde N(x,dy))$ of X reads as follows:

$$
\begin{aligned}
\tilde\ell^{(1)}(x) = & \Big(\ell\exp(\tfrac{x^{(2)}}{2})\Big)\\
& +\Big(\exp(\tfrac{x^{(2)}}{2})\int_{\mathbb{R}\setminus\{0\}} y\cdot(1_{\left\{\left|\exp(\frac{x^{(2)}}{2})y\right|<1\right\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}})\ N(dy)\Big)\\
\tilde\ell^{(2)}(x) = & \Big(\tfrac{\beta}{\exp(x^{(2)})}+\log\delta\Big)\\
& +\Big(\int_{\mathbb{R}\setminus\{0\}}\log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\left\{\left|\exp(x^{(2)}/2)y\right|<1\right\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}})\ N(dy)\Big)\\
\tilde Q(x) = & \begin{pmatrix}\exp(x^{(2)})Q & 0\\ 0 & 0\end{pmatrix}\\
\tilde N(x,dy) = & f_*^{x^{(2)}}N(dy).
\end{aligned}
$$

Proof: In order to keep the formulas readable we write

$$(G,V)' = (X^{(1)},X^{(2)})' \qquad \text{and} \qquad (g,v)' = (x^{(1)},x^{(2)})'.$$

We use Theorem 4.4 to calculate the symbol. T is the first entrance time of the complement of the compact set $\overline{B_0(R)}$. At first we use Itô's formula and the fact that G is homogeneous in space:

$$
\begin{aligned}
\frac{\mathbb{E}^{g,v} e^{i(G_t^T - g, V_t^T - v)\xi} - 1}{t} &= \frac{\mathbb{E}^{0,v} e^{i(G_t^T, V_t^T - v)\xi} - 1}{t} \\
&= \frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^{t} i\xi^{(1)} e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, dG_s^T \\
&\quad + \frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^{t} i\xi^{(2)} e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, dV_s^T \\
&\quad - \frac{1}{2t}\mathbb{E}^{0,v} \int_{0+}^{t} i(\xi^{(1)})^2 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, d[G^T, G^T]_s^c \\
&\quad - \frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^{t} i\xi^{(1)}\xi^{(2)} e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, d[G^T, V^T]_s^c \\
&\quad - \frac{1}{2t}\mathbb{E}^{0,v} \int_{0+}^{t} i(\xi^{(2)})^2 e^{i(G_{s-}^T, V_{s-}^T - v)\xi}\, d[V^T, V^T]_s^c \\
&\quad + \frac{1}{t}\mathbb{E}^{0,v} \sum_{0<s\le t} e^{(G_{s-}^T, V_{s-}^T - v)\xi} \left(e^{i\Delta(G_s^T, V_s^T)\xi} - 1 - (i\xi^{(1)}\Delta G_s^T + i\xi^{(2)}\Delta V_s^T) \right) \\
&=: I_1 + I_2 - I_3 - I_4 - I_5 + I_6.
\end{aligned}
$$

We deal with this formula term-by-term. In the calculation of the first term (I_1) we use

$$dG^T = \sigma_{s-} 1_{[[0,T]]}(s)\ dZ_s.$$

Let us remark that the integrand is bounded and for the Lévy process Z we have the Lévy-Itô-decomposition [8]:

$$
\begin{aligned}
Z_t \;=\; & \underbrace{\sqrt{Q}W_t}_{\text{Gaussian part}} + \underbrace{\int_{[0,t]\times\{|y|<1\}} y\ (\mu^Z(ds,dy) - ds N(dy))}_{\text{compensated small jumps}} && L^2\text{-martingale} \\
& + \underbrace{\ell t}_{\text{drift}} + \underbrace{\sum_{0<s\le t} \Delta Z_s 1_{\{|\Delta Z_s|\ge 1\}}}_{\text{big jumps}} && \text{finite variation.}
\end{aligned}
$$

The integrals with respect to the martingale parts are again L^2-martingales and the respective terms disappear. What remains from I_1 is

$$\frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^{t} i\xi^{(1)} e^{i(G_{s-}^T, V_{s-}^T - v)\xi} \sigma_{s-} 1_{[[0,T]]}(s)\ d\left(\ell s + \sum_{0<r\le s} \Delta Z_r \cdot 1_{\{|\Delta Z_r|\ge 1\}}\right). \tag{7.3}$$

For the first part of this integrand we get

$$
\begin{aligned}
&\frac{1}{t}\mathbb{E}^{0,v} \int_{0+}^{t} i\xi^{(1)} e^{i(G_{s-}^T, V_{s-}^T - v)\xi} \sigma_{s-} 1_{[[0,T]]}(s)\ d(\ell s) \\
&\quad = \mathbb{E}^{0,v} \frac{1}{t} \int_0^t i\xi^{(1)} \ell e^{i(G_s^T, V_s^T - v)\xi} 1_{[[0,T[[}(s) \sigma_s\ ds \\
&\quad = i\xi^{(1)}\ell\ \mathbb{E}^{0,v} \frac{1}{t} \int_0^t \underbrace{e^{i(G_s^T, V_s^T - v)\xi} 1_{[[0,T[[}(s)}_{\to 1} \underbrace{\sigma_s}_{\to \widetilde{s}}\ ds \\
&\underset{t\downarrow 0}{\longrightarrow} \quad i\xi^{(1)}\ell\widetilde{s}
\end{aligned}
$$

[8] We write $\sqrt{Q}$ instead of σ because the latter one could be mixed up with the volatility process.

where $\widetilde{s}$ is the starting point of the process σ (see above). Here we used the fact that we are integrating with respect to Lebesgue measure. For this the countable number of jumptimes is a nullset. In the last step we used Lemma 2.51 a). A similar argumentation is used in the consideration of I_2 and I_3. The jump-term of (7.1) above will be compared to I_6.

Using Itô's Formula we obtain for the second term

$$\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi^{(2)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\frac{1}{\sigma_{s-}^2}\left\{d(\sigma_s^T)^2+d\left(\sum_{0<r\leq s}\log\sigma_r^2-\log\sigma_{r-}^2-\frac{1}{\sigma_{r-}^2}\Delta(\sigma_r^2)\right)\right\}$$

and by plugging in the defining SDE for $(\sigma_t^2)_{t\geq 0}$

$$\begin{aligned}&\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi^{(2)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{[[0,T]]}(s)\ \Bigg\{\left(\frac{\beta}{\sigma_{s-}^2}\ ds+\frac{\sigma_{s-}^2}{\sigma_{s-}^2}\log\delta\ ds\right)\\&+\frac{\lambda}{\delta}\ d\left(\sum_{0<r\leq s}(\Delta Z_r)^2)\right)+d\left(\sum_{0<r\leq s}\Delta(\log\sigma_r^2)-\frac{1}{\sigma_{r-}^2}\Delta(\sigma_r^2)\right)\Bigg\}.\end{aligned}$$

We postpone the treatment of the jump parts and for the remainder term we get, using a similar argumentation as for the first term

$$i\xi^{(2)}\beta/\widetilde{s}^2+i\xi^{(2)}\log\delta.$$

For I_3 we obtain in an analogous manner to I_1:

$$\begin{aligned}&-\frac{1}{2t}\mathbb{E}^{0,v}\int_{0+}^{t} i(\xi^{(1)})^2e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\ d[G^T,G^T]_s^c\\&\quad=-\frac{1}{2t}\mathbb{E}^{0,v}\int_{0+}^{t} i(\xi^{(1)})^2e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\ 1_{[[0,T]]}(s)\sigma_{s-}^2\ d[Z,Z]_s^c\\&\quad=-\frac{1}{2t}\mathbb{E}^{0,v}\int_{0+}^{t} i(\xi^{(1)})^2e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\ 1_{[[0,T]]}(s)\sigma_{s-}^2\ d(Qs)\\&\to-\frac{1}{2}(\xi^{(1)})^2\widetilde{s}^2Q.\end{aligned}$$

The terms four and five are constantly zero: Since $t\longmapsto t$ and $t\longmapsto[Z,Z]_t$ are both of finite variation on compacts the process $(\sigma_t^2)_{t\geq 0}$ has this property as well, by the very definition. Therefore, it is a quadratic pure jump process (see Definition 2.37). Using Itô's Formula we obtain that $V=\log(\sigma^2)$ is again a quadratic pure jump process and therefore

$$[V^T,V^T]_s^c=0\text{ and }[V^T,G^T]_s^c=0.$$

The only thing that remains to do is dealing with the various 'jump parts'. From the first term we left the following behind:

$$\begin{aligned}&\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi^{(1)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\sigma_{s-}1_{[[0,T]]}(s)\ d\left(\sum_{0<r\leq s}\Delta Z_r\cdot 1_{\{|\Delta Z_r|\geq 1\}}\right)\\&\quad=\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\leq t} i\xi^{(1)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}\sigma_{s-}1_{[[0,T]]}(s)\Delta Z_s\cdot 1_{\{|\Delta Z_s|\geq 1\}}.\end{aligned}$$

And from the second one:

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi^{(2)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{[[0,T]]}(s)\frac{\lambda}{\delta}\;d\left(\sum_{0<r\le s}(\Delta Z_r)^2\right)\\
&\qquad+\frac{1}{t}\mathbb{E}^{0,v}\int_{0+}^{t} i\xi^{(2)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{[[0,T]]}(s)\;d\left(\sum_{0<r\le s}\Delta V_r-\frac{1}{\sigma_{r-}^2}\Delta(\sigma_r^2)\right)\\
&=\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t} i\xi^{(2)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{[[0,T]]}(s)\frac{\lambda}{\delta}(\Delta Z_s)^2\\
&\qquad+\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t} i\xi^{(2)}e^{i(G_{s-}^T,V_{s-}^T-v)\xi}1_{[[0,T]]}(s)\left(\Delta V_s-\frac{1}{\sigma_{s-}^2}\Delta(\sigma_s^2)\right).
\end{aligned}$$

Adding these terms to I_6 and using the equalities

$$\Delta G_s^T=(\sigma_{s-}1_{[[0,T]]}(s))\Delta Z_s \text{ and } (\Delta\sigma_s^T)^2=\frac{\lambda}{\delta}(\sigma_{s-}^2 1_{[[0,T]]}(s))(\Delta Z_s)^2$$

and

$$\Delta\log(\sigma_s^2)^T=\log\left(\frac{(\sigma_{s-}^2)^T+\Delta(\sigma_s^2)^T}{(\sigma_{s-}^2)^T}\right)=\log\left(1+\frac{\Delta(\sigma_s^2)^T}{(\sigma_{s-}^2)^T}\right)$$

we obtain

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^{0,v}\sum_{0<s\le t}\exp(i(G_{s-}^T,V_{s-}^T-v)\xi)1_{[[0,T]]}(s)\;\times\\
&\qquad\big(\exp(i\sigma_{s-}\Delta Z_s\xi^{(1)}+i\log(1+(\lambda/\delta)\Delta(Z_s)^2)\xi^{(2)})-1-i\xi^{(1)}\sigma_{s-}\Delta Z_s\cdot 1_{\{|\Delta Z_s|<1\}}\big)\\
&=\frac{1}{t}\mathbb{E}^{0,v}\int_{]0,t]\times\{y\neq 0\}}\exp(i(G_{s-}^T,V_{s-}^T-v)\xi)1_{[[0,T]]}(s)\;\times\\
&\qquad\big(\exp(i\sigma_{s-}y\xi^{(1)}+i\log(1+(\lambda/\delta)y^2)\xi^{(2)})-1-i\xi^{(1)}\sigma_{s-}y\cdot 1_{\{|y|<1\}}\big)\;\mu^Z(\cdot;ds,dy)\\
&=\frac{1}{t}\mathbb{E}^{0,v}\int_{]0,t]\times\{y\neq 0\}}\exp(i(G_{s-}^T,V_{s-}^T-v)\xi)1_{[[0,T]]}(s)\;\times\\
&\qquad\Bigg(\bigg(\exp(i\sigma_{s-}y\xi^{(1)}+i\log(1+(\lambda/\delta)y^2)\xi^{(2)})-1\\
&\qquad-i\Big((\sigma_{s-}y)\xi^{(1)}+(\log(1+\tfrac{\lambda}{\delta}y^2))\xi^{(2)}\Big)\cdot 1_{\{|\tilde{s}y|<1\}}\cdot 1_{\{|\log(1+\frac{\lambda}{\delta}y^2)|<1\}}\bigg)\\
&\qquad+\Big(i\xi^{(1)}\sigma_{s-}y\cdot(1_{\{|\tilde{s}y|<1\}}\cdot 1_{\{|\log(1+\frac{\lambda}{\delta}y^2)|<1\}})-1_{\{|y|<1\}})\Big)\\
&\qquad+\Big(i\xi^{(2)}\log(1+\tfrac{\lambda}{\delta}y^2)\cdot 1_{\{|\tilde{s}y|<1\}}\cdot 1_{\{|\log(1+\frac{\lambda}{\delta}y^2)|<1\}}\Big)\Bigg)\;\mu^Z(\cdot;ds,dy).
\end{aligned}$$

It is possible to calculate the integral with respect to the compensator $\nu(\cdot; ds, dy) = N(dy)\, ds$ instead of the measure itself 'under the expectation', since the integrands are of class F_p^1 of Ikeda and Watanabe (see Theorem 2.21 a)). One obtains this, because

$$1_{\{|\widetilde{s}y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}}$$

is zero near the origin and bounded and

$$\log(1+\frac{\lambda}{\delta}y^2) \leq (\lambda/\delta)\cdot y^2 \qquad \text{for} \qquad \left|\frac{\lambda}{\delta}\cdot y^2\right| < 1.$$

Letting t tend to zero (and multiplying with -1) we obtain by using Lemma 2.51 a)

$$\begin{aligned}
&p\left(\begin{pmatrix} g \\ v \end{pmatrix}, \xi\right) = \\
&\quad -i\xi^{(1)}\left(\ell\widetilde{s} + \widetilde{s}\int_{\mathbb{R}\backslash\{0\}} y\cdot\left(1_{\{|\widetilde{s}y|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}}\right)\, N(dy)\right) \\
&\quad -i\xi^{(2)}\left(\frac{\beta}{\widetilde{s}^2} + \log\delta + \int_{\mathbb{R}\backslash\{0\}} \log(1+\frac{\lambda}{\delta}y^2)\cdot\left(1_{\{|\widetilde{s}y|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}}\right)\, N(dy)\right) \\
&\quad +\frac{1}{2}(\xi^{(1)})^2\widetilde{s}^2 Q \\
&\quad -\int_{\mathbb{R}^2\backslash\{0\}} \left(\exp(i(z_1,z_2)\xi) - 1 - iz'\xi\cdot\left(1_{\{|z_1|<1\}}\cdot 1_{\{|z_2|<1\}}\right)\right) h_*^{\widetilde{s}}N\,(dz)
\end{aligned}$$

where $h_*^{\widetilde{s}}N$ is the image measure

$$h_*^{\widetilde{s}}N\,(B) := N(h^{\widetilde{s}} \in B) \qquad \text{for} \qquad B \in \mathcal{B}^2$$

and the function $h^{\widetilde{s}} : \mathbb{R} \longrightarrow \mathbb{R}^2$ is given by

$$h^{\widetilde{s}} : \eta \longmapsto \begin{pmatrix} \widetilde{s}\eta \\ \log(1+(\lambda/\delta)\ \eta^2) \end{pmatrix} \qquad \text{for} \qquad \widetilde{s} \in \mathbb{R}.$$

And by writing the starting point as $(x^{(1)}, x^{(2)})' = (g,v)' = (g, 2\log(\widetilde{s}))$ we obtain the result since $h^{\exp(v/2)} = f^v = f^{x^{(2)}}$. $\square$

Using our results of Section 6.1 we obtain at once the (extended) generator of the process $(X^{(1)}, X^{(2)})' = (G, \log(\sigma^2))'$ with starting point $(x^{(1)}, x^{(2)})'$.

$$
\begin{aligned}
A_{ext}u(x) &= \partial_1 u(x)\Big(\ell \exp(x^{(2)}/2)\Big)\\
&+\partial_1 u(x)\Big(\exp(x^{(2)}/2) \int_{\mathbb{R}\backslash\{0\}} y \cdot (1_{\{|\exp(x^{(2)}/2)y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}})\ N(dy)\Big)\\
&+\partial_2 u(x)\Big(\frac{\beta}{\exp(x^{(2)})} + \log\delta\Big)\\
&+\partial_2 u(x)\Big(\int_{\mathbb{R}\backslash\{0\}} \log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\{|\exp(x^{(2)}/2)y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}})\ N(dy)\Big)\\
&+\partial_1\partial_1 u(x)\exp(x^{(2)})Q\\
&+\int_{\mathbb{R}^2\backslash\{0\}} \big(u(x-y)-u(x)+y'\nabla u(x)\cdot(1_{\{|y_1|<1\}}\cdot 1_{\{|y_2|<1\}})\big)\, f_*^{x^{(2)}} N\,(dy)
\end{aligned}
$$

for $u \in C_b^2$ with the $f_*^{x^{(2)}}N$ from (7.1). The semimartingale characteristics of $(X^{(1)}, X^{(2)})'$ are

$$
\begin{aligned}
B_t^{(1)} &= \int_0^t \Big(\ell\exp(\frac{X^{(2)}}{2})\\
&\quad + \exp(\tfrac{X^{(2)}}{2}) \int_{\mathbb{R}\backslash\{0\}} y\cdot(1_{\{|\exp(\frac{X^{(2)}}{2})y|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}} - 1_{\{|y|<1\}})\ N(dy)\Big)\ ds\\
B_t^{(2)} &= \int_0^t \Big(\frac{\beta}{\exp(X^{(2)})} + \log\delta\\
&\quad + \int_{\mathbb{R}\backslash\{0\}} \log(1+\frac{\lambda}{\delta}y^2)\cdot(1_{\{|\exp(X^{(2)}/2)y|<1\}}\cdot 1_{\{|\log(1+(\lambda/\delta)\ y^2)|<1\}})\ N(dy)\Big)\ ds\\
C_t &= \int_0^t \begin{pmatrix} \exp(X^{(2)})Q & 0\\ 0 & 0\end{pmatrix} ds\\
\nu(\cdot; ds, dy) &= f_*^{X_s(\cdot)^{(2)}} N(dy)\ ds.
\end{aligned}
$$

Let us remark that the generator and the characteristics of the process (G, σ^2) have been calculated in the recent paper [39]. The authors use some abstract results of [37] and results of the calculus of semimartingale characteristics. The 'truncation function' which is used in the paper for two-dimensional processes is neither a cut-off function nor a truncation function in the sense of Section 1.1.

7.2 Ornstein-Uhlenbeck Process, Revisited

The Ornstein-Uhlenbeck process is the solution of the following SDE (see [40] Example 5.6.8):

$$\begin{aligned} dX_t &= -\alpha X_t \, dt + \sigma \, dW_t \\ X_0 &= x \end{aligned}$$

for $\alpha, \sigma > 0$, $x \in \mathbb{R}$ and a standard Brownian motion W.

This process is well understood and we give only a short account on how this process can be seen from our perspective: Since every deterministic process is independent of W, $(t, W_t)'_{t \geq 0}$ is a Lévy process (see Proposition 1.26 b)). Therefore, we are in the setting of Section 5.1 ($n = 2$ and $d = 1$) with

$$\Phi(x) = (-\alpha x, \sigma) \qquad \text{and} \qquad \psi(\eta) = -i\eta^{(1)} + \left|\eta^{(2)}\right|^2 .$$

We obtain from Theorem 5.1 that the symbol of the solution is

$$p(x, \xi) = \psi(\Phi(x)'\xi) = i\alpha x \xi + |\sigma \xi|^2 .$$

A The Fine Topology

In Chapter 4 the so called 'fine topology' turned out to be the best possible condition under which the existence of the probabilistic symbol could be proved. In this section we mainly follow [9]. For some more information on this topic see [20]. We fix a (universal) Markov process

$$\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, (\vartheta_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}.$$

In particular we have a family of measures $\mathbb{P}^\mu$ by the formula

$$\mathbb{P}^\mu(B) = \int_{\mathbb{R}^d} \mathbb{P}^x(B)\, d\mu(x) \quad \text{for every } B \in \mathcal{F}_\infty$$

where μ is a probability measure on the state space. First we introduce the following concept:

Definition A.1 A set $A \subset \mathbb{R}^d$ (or more generally $\mathbb{R}^d_\Delta$) is called **nearly Borel set** with respect to the given process X, if for each initial measure μ there exists B and B' in $\mathcal{B}^d$ (resp. $\mathcal{B}^d_\Delta$) such that $B \subset A \subset B'$ and

$$\mathbb{P}^\mu(X_t \in B' \backslash B \text{ for some } t \geq 0) = 0.$$

Roughly speaking a set is nearly Borel, if the process cannot distinguish it from a Borel set. The class of nearly Borel sets is a σ-algebra in $\mathbb{R}^d$ (resp. in $\mathbb{R}^d_\Delta$). We denote it by $(\mathcal{B}^d)^n$ (resp. $(\mathcal{B}^d_\Delta)^n$. Obviously we have

$$\mathcal{B}^d \subset (\mathcal{B}^d)^n \subset (\mathcal{B}^d)^*$$

and a similar relationship holds for $\mathbb{R}^d_\Delta$. Nearly Borel measurable functions are characterized by the property that for each μ there exist Borel measurable functions f_1, f_2 such that $f_1 \leq f \leq f_2$ and

$$\mathbb{P}^\mu(f_1(X_t) \neq f_2(X_t) \text{ for some } t \geq 0) = 0.$$

Let $T_D(\omega) := \inf\{t > 0 : X_t(\omega) \in D\}$ denote the **first hitting time** of D.

Definition A.2 A set $A \subset \mathbb{R}^d_\Delta$ is called **finely open**, if for every $x \in A$ there is a set $D \in (\mathcal{B}^d_\Delta)^n$ such that $A^c \subset D$ and $\mathbb{P}^x(T_D > 0) = 1$.

Intuitively a set A is finely open provided the process remains in A for an initial interval of time almost surely $\mathbb{P}^x$ for each $x \in A$. By the right continuity of the paths any open set is finely open, i.e. the fine topology given by these sets is finer than the original topology on $\mathbb{R}^d_\Delta$. We do not have to go into details here. What we need is the following theorem:

Theorem A.3 *Suppose f is nearly Borel measurable. Then f is finely continuous if and only if the mapping $t \longmapsto f(X_t)$ is right continuous almost surely.*

Proof: See [9] Theorem II.4.8. □

B (Counter-)examples

In this section we give some examples of stochastic processes which are useful in this context. For the sake of the readability of the previous chapters they were put here.

Example B.1 (a pre-stopped martingale) In Theorem 2.15 we have seen, that the notion of pre-stopping is compatible with semimartingales. Now we investigate the one-dimensional martingale $M = (M_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P}) := (\{0,1\}, \{\emptyset, \{0\}, \{1\}, \Omega\}, 1/2 \cdot (\delta_0 + \delta_1)$ given by

$$M_t(\omega) = (-1)^{\omega+1} 1_{[1,\infty[}(t)$$

for every $t \geq 0$. If we pre-stop this process with the first entrance time of the set $\{-1\}$ which we denote by S, we obtain

$$M_t^{S-}(\omega) = \omega \cdot 1_{[1,\infty[}(t).$$

This process is not a martingale, since $\mathbb{E} M_0^{S-} = 0 \neq 1/2 = \mathbb{E} M_2^{S-}$ (see 0.35). It is not even a local martingale: for every sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ such that $T_n \uparrow \infty$ there exists a $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $T_n \geq 2$ and therefore, $M^{S-} = (M^{S-})^{T_n}$.

Example B.2 (a solution of a 'Lévy SDE' which is not time-homogeneous Markovian) In Theorem 2.47 we have seen, that the solution of the SDE (2.10), which is driven by a d-dimensional Lévy process is a time-homogeneous Markov process. It is a natural question, whether this statement remains true, if we replace the driving process by an arbitrary vector of one-dimensional Lévy processes:
Let W, U be independent one-dimensional Brownian motions and $T := 1$ a deterministic stopping time. Set $\widetilde{W} := -(W^T + U \circ \vartheta_{-1})$, where $(U \circ \vartheta_{-1})_t(\omega) = U_{t-1}(\omega) \cdot 1_{[1,\infty[}(t)$. Using some well known facts on Brownian motion (see e.g. [40]) one obtains, that $\widetilde{W}$ is again a standard Brownian motion. Consider the SDE:

$$X_t = 1 + \int_0^t X_s \, dW_s + \int_0^t X_s \, d\widetilde{W}_s = \begin{cases} 1 & \text{if } t \leq 1 \\ 1 + \int_0^t X_s \, d(W_s - W_s^T + (U \circ \theta_{-1})_s) & \text{if } t > 1 \end{cases}$$

The solution of this SDE is 1 on $[0,1[$ and a (shifted) stochastic exponential on $[1,\infty[$ (see [50] Section II.8). This solution is not time-homogeneous, since we have

$$P_{0,1}(1, \{1\}) = 1 \neq P_{2,3}(1, \{1\}).$$

Example B.3 (a solution of a 1-dimensional Lévy driven SDE which is not a Feller process) In Theorem 2.49 we have seen that the solution of the Lévy driven SDE

$$X_t = x + \int_0^t \Phi(X_{s-}) \, dZ_s$$

is a Feller process, if Φ is bounded and Lipschitz continuous. The present example shows, that the solution is in general not a Feller process, if Φ is not bounded. Consider the SDE

$$X_t = x - \int_0^t X_{s-} \, dN_s$$

where $N = (N_t)_{t\geq 0}$ is standard Poisson process (see Example 2.40). The solution starts in x, stays there for an exponentially distributed time (which is independent of x) and then jumps to zero, where it remains. This is in line with the formula for the stochastic exponential (see e.g. [50] Theorem II.37). Since the waiting time (in x) is exponentially distributed, there exists a time $t_0 > 0$ for which $\mathbb{P}^x(X_{t_0} = x) = \mathbb{P}^x(X_{t_0} = 0) = 1/2$. For a function $u \in C_c(\mathbb{R})$ with the property $u(0) = 1$ we obtain

$$\mathbb{E}^x(u(X_{t_0})) = \frac{1}{2} \qquad \text{for every } x \notin \text{ supp } u.$$

Therefore, $T_{t_0}u$ does not vanish at infinity. The symbol of this process is

$$p(x,\xi) = \lambda(1 - \exp(-ix\xi)).$$

Let us remark that although the solution is not a Feller process in the sense of Definition 1.19 it is a C_b-Feller process which is sometimes considered in the literature. Compare in this context [54].

Example B.4 (a Hunt process which is not a semimartingale) Let W be a one-dimensional standard Brownian motion and define $X := |W|^{1/2}$. In [59] it is shown that X is a Hunt process but not a semimartingale.

Example B.5 (a Hunt semimartingale which is not an Itô process) Let W be a one-dimensional standard Brownian motion. The process $X := |W|$ is a Hunt semimartingale additive functional but not an Itô process. Compare in this context [12] Example (3.58).

Example B.6 (an Itô process which is not a Feller process) Consider the transition semigroup of the 'space dependent drift'

$$P_t(x,B) = \begin{cases} 1_{B-t}(x) & \text{if } x > 0 \\ 1_B(x) & \text{if } x = 0 \\ 1_{B+t}(x) & \text{if } x < 0, \end{cases}$$

where $t \geq 0$, $x \in \mathbb{R}$ and $B \in \mathcal{B}$. Obviously this corresponds to

$$T_t u(x) = \mathbb{E}^x u(X_t) = \begin{cases} u(x+t) & \text{if } x > 0 \\ u(0) & \text{if } x = 0 \\ u(x-t) & \text{if } x < 0 \end{cases}$$

for $u \in C_\infty(\mathbb{R})$. The semigroup $(T_t)_{t\geq 0}$ is not Feller: we obtain for $u \in C_\infty(\mathbb{R})$ with $u(0) = 2$ and $u(1) = 0$ that $T_1(0+) = u(1) = 0$ but $T_1(0) = u(0) = 2$. Therefore, $T_1 u$ is not continuous. But since the corresponding process X is of finite variation it is a semimartingale and even an Itô process with characteristics $(B, C, \nu) = (X - x, 0, 0)$ where B can be written as

$$B_t = \text{sign}(x) \cdot t = \begin{cases} t & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -t & \text{if } x < 0 \end{cases}$$

or

$$B_t = \int_0^t \operatorname{sign}(X_s)\ ds$$

i.e. $\ell(x) = \operatorname{sign}(x)$. The symbol of this process is

$$p(x,\xi) = -i \cdot \operatorname{sign}(x)\xi.$$

This symbol is not continuous in x, but it is finely continuous, since $t \longmapsto \ell(X_t)$ is right continuous for every $\mathbb{P}^x$ $(x \in \mathbb{R})$.

Example B.7 (a Feller process which is not a Lévy process) In Section 2.6 we have shown that the solution of the Lévy driven SDE

$$X_t = x + \int_0^t \Phi(X_{s-})\ dZ_s$$

where Φ is bounded and Lipschitz continuous is a nice Feller process. However, since the symbol

$$p(x,\xi) = \psi(\Phi(x)'\xi)$$

depends on $x \in \mathbb{R}^d$ the process is not Lévy.

C Friedrichs mollifier

We need the Friedrichs mollifier as a tool to approximate $C_c^2(\mathbb{R}^d)$-functions:
Let $\psi_0(t) := 1_{]-\infty,0[}(t) \cdot \exp(\frac{1}{t})$. This is a function in $C^\infty(\mathbb{R})$. Furthermore, we set: $\psi(x) := \psi_0(\|x\|^2 - 1) \in C_c^\infty(\mathbb{R}^d)$. The support of ψ is contained in $\overline{B_1(0)}$.

Theorem C.1 *Let $f \in C_c^k(\mathbb{R}^d)$ with $0 \le k \le \infty$ and let $\rho \in C_c^\infty(\mathbb{R}^d)$ such that*

$$\rho \ge 0,\ \text{supp}\ \rho \subset \overline{B_1(0)},\ \int \rho(x)\ dx = 1.$$

Let $\varepsilon > 0$ and

$$f_\varepsilon(x) := \varepsilon^{-n} \int f(y)\rho\left(\frac{x-y}{\varepsilon}\right)\ dy$$

then $f_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ and $\text{supp}\ f \subset \text{supp}\ f_\varepsilon + \overline{B_0(\varepsilon)}$. *And:*

$$\textit{for } |\alpha| \le k \textit{ we have } \partial^\alpha f_\varepsilon \xrightarrow[\varepsilon\to 0]{\|\cdot\|_\infty} \partial^\alpha f.$$

Remark: One can use $\rho := \psi / \int \psi(x)\ dx$.

Proof: See [18] page 6. □

We will use the next corollary which can be easily obtained, by setting $\varepsilon := \delta/n$ and $k = 2$ in the theorem above.

Corollary C.2 *For $f \in C_c^2(\mathbb{R}^d)$ and $\delta > 0$ there exists a sequence $f_n \in C_c^\infty(\mathbb{R}^d)$ such that*

$$f_n \xrightarrow[n\to\infty]{\sum_{|\alpha|\le 2}\|\partial^\alpha\cdot\|_\infty} f \qquad \textit{and} \qquad \text{supp}\ f_n \subset \text{supp}\ f + \overline{B_0(\delta)}.$$

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Index of Notation

Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor in any other country.
The present thesis has been written under the supervision of Prof. Dr. René L. Schilling from January 2006 to December 2008. From January 1, 2006 to July 31, 2008 the author has been employed at the Philipps-Universität Marburg. Since August 1, 2008 he is working at the Faculty of Science, TU Dresden.
There have been no prior attempts to obtain a PhD at any other university.
I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued March 20, 2000 with the changes in effect since April 16, 2003.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.
Diese Arbeit wurde angefertigt unter der Betreuung von Prof. Dr. René L. Schilling in der Zeit von Januar 2006 bis Dezember 2008. Der Verfasser war vom 1. Januar 2006 bis zum 31. Juli 2008 an der Philipps-Universität Marburg als wissenschaftlicher Mitarbeiter auf Zeit beschäftigt. Seit dem 1. August 2008 hat er eine entsprechende Stelle an der TU Dresden inne.
Es wurden zuvor keine Promotionsvorhaben unternommen.
Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 20. März 2000, in der geänderten Fassung mit Gültigkeit vom 16. April 2003, an.

Dresden, 15. Dezember 2008

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